# Week 7

151WIRH7

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# MATH 151- WEEK-IN-REVIEW 7

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## EXAM 2 REVIEW

1. Find the point(s) on the curve $x = t^2 + 4t$, $y = t^2 + 5t$ where the tangent line is vertical or horizontal.

$$\frac{dy}{dx} = \frac{2t+5}{2t+4}$$

Horizontal $t = -\frac{5}{2}$

$x(-\frac{5}{2}) = (-\frac{5}{2})^2 + 4(-\frac{5}{2}) = \frac{25}{4} - 10 = -\frac{15}{4}$

$y(-\frac{5}{2}) = (-\frac{5}{2})^2 + 5(-\frac{5}{2}) = \frac{25}{4} - \frac{25}{2} = -\frac{25}{4}$

$\boxed{\left(-\frac{15}{4}, -\frac{25}{4}\right)}$ horizontal

Vertical $2t+4=0$, $t=-2$

$x(-2) = (-2)^2 + 4(-2) = -4$

$y(-2) = (-2)^2 + 5(-2) = -6$

$\boxed{(-4,-6)}$ Vertical

2. Find the tangent vector of unit length for $\vec{r}(t) = \langle e^{t^2}, 3t\cos(t)\rangle$ at $t = 0$.

(a) $\langle 0, 1\rangle$

(b) $\langle 0, -3\rangle$

(c) $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$

(d) $\langle 0, -1\rangle$

(e) $\left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\rangle$

take derivative

plug in $t=0$

divide by magnitude

$r'(t) = \langle e^{t^2}\cdot 2t,\ 3t(-\sin t) + 3\cos(t)\rangle$

$r'(0) = \langle 0, 3\rangle$

$\|r'(0)\| = \sqrt{0^2+3^2}$

$\frac{r'(0)}{\|r'(0)\|} = \frac{\langle 0,3\rangle}{3} = \langle 0,1\rangle$

3. The radius of a sphere was measured to be 10in with a possible error of 0.25in. Use differentials to estimate the maximum error in the calculated surface area and find the relative error.

$\boxed{S = 4\pi r^2}$

$dS = 8\pi r \cdot dr$

$= 8\pi(10)\cdot 0.25$

$= \boxed{20\pi \text{ in}^2}$

Relative: $\frac{\text{error}}{\text{measured}} = \frac{20\pi}{4\pi(10)^2} = \frac{20\pi}{4\pi\cdot 100} = \frac{20\pi}{400\pi} = \boxed{\frac{1}{20}}$

4. If $H(x) = f(g(x^2 + 4x))$, find $\boxed{H'(1)}$ given that $f'(1) = 2$, $f'(5) = 0$, $g(5) = 1$, $g'(1) = 4$, and $g'(5) = 3$.

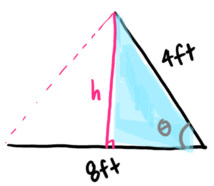
(a) 36
(b) 2
(c) 0
(d) $-8$
(e) $-30$

$$H'(x) = f'(g(x^2+4x)) \cdot g'(x^2+4x) \cdot (2x+4)$$

$$H'(1) = f'(g(5)) \cdot g'(5) \cdot (6)$$
$$= f'(1) \cdot 3 \cdot 6$$
$$= 2 \cdot 3 \cdot 6$$
$$= 36$$

5. Two sides of a triangle have length 8ft and 4ft. The angle between them at a rate of $\frac{\pi}{8}$ rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\frac{\pi}{3}$.

increases




$A = \frac{1}{2}bh$ base is fixed

$A = \frac{1}{2} 8 \cdot h = 4h$

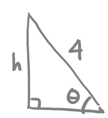
height is changing (variable)

$\frac{dA}{dt} = 4 \frac{dh}{dt}$ ?

$\frac{dA}{dt} = 4 \cdot \frac{\pi}{4}$

$= \boxed{\pi \ ft^2/s}$

$\frac{dh}{dt}$




differentiate

$\sin(\theta) = \frac{h}{4} = \frac{1}{4}h$

$\cos(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{4}\frac{dh}{dt}$

$\cos\left(\frac{\pi}{3}\right) \cdot \frac{\pi}{8} = \frac{1}{4}\frac{dh}{dt}$

$4 \cdot \frac{1}{2} \cdot \frac{\pi}{8} = \frac{dh}{dt}$

$\frac{\pi}{4} = \frac{dh}{dt}$

6. Find the equation of the tangent line to the curve $2x^3y - 5y^4 = 11$ at the point $(2, 1)$.

$$2x^3 \frac{dy}{dx} + 6x^2y - 20y^3\frac{dy}{dx} = 0$$

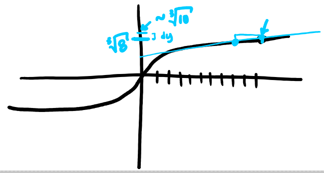
$$2x^3 \frac{dy}{dx} - 20y^3\frac{dy}{dx} = -6x^2y$$

$$\frac{dy}{dx}(2x^3 - 20y^3) = -6x^2y$$

$$\frac{dy}{dx} = \frac{-6x^2y}{2x^3 - 20y^3}$$

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-6(4)(1)}{2(8) - 20(1)} = \frac{-24}{16-20} = 6$$

$$\boxed{y - 1 = 6(x-2)}$$




7. Using differentials or a linear approximation, approximate $\sqrt[3]{10}$.

Using differentials

$y = (x)^{1/3}$ $a = 8$

$dy = \frac{1}{3}x^{-2/3}\,dx$

at $a = 8$ $dx = 10 - 8$

$\sqrt[3]{8^2} = \sqrt[3]{64}$

$(\sqrt[3]{8})^2 = (2)^2$

$dy = \boxed{\frac{1}{3\,(8)^{2/3}}}(2) = \frac{1}{3\cdot 4}(2) = \frac{1}{6}$

$y - y_1 = m(x - x_1)$

$y = \left(\frac{1}{3}x_1^{-2/3}\right)(x - 8) + \sqrt[3]{8}$

$= \frac{1}{12}(x - 8) + \sqrt[3]{8}$

$= \frac{1}{12}(2) + \sqrt[3]{8}$

Approximation: $\sqrt[3]{8} + \frac{1}{6} = 2 + \frac{1}{6} = \boxed{\frac{13}{6}}$

8. A particle moves according to the equation $s(t) = t^2 - 4t + 1$ where $t$ is measured in seconds and $s$ is in feet. What is the total distance the particle travels in the first 3 seconds?

$S(t) = t^2 - 4t + 1$

$v(t) = 2t - 4$

$0 = 2t - 4$

$t = 2$

4 ft 1 ft

$t=0$ $t=2$ $t=3$

$s(0) = 1$ $s(2) = -3$ $s(3) = -2$

$\boxed{\text{5 feet}}$

$\boxed{\text{displacement} = -3 \text{ ft}}$

9. Calculate the $77^{th}$ derivative of $g(x) = 2\sin(4x)$.

~~(a) $g^{(77)} = 2^{78}\cos(4x)$~~
~~(b) $g^{(77)} = -2^{78}\sin(4x)$~~
(c) $g^{(77)} = 2^{155}\cos(4x)$
(d) $g^{(77)} = -2^{155}\cos(4x)$
(e) $g^{(77)} = 2^{155}\sin(4x)$

$g'(x) = 2\cos(4x)\cdot 4$

$g''(x) = -2\sin(4x)\cdot 4\cdot 4$

$g'''(x) = -2\cos(4x)\cdot 4\cdot 4\cdot 4$

76th

$g^{(4)}(x) = 2\sin(4x)\cdot 4\cdot 4\cdot 4\cdot 4$

$(4)^{77} = (2^2)^{77} = 2^{154}$

$2^{154+1} = 2^{155}$

$4\overline{)77} = 19$
$-4$
$37$
$-36$
$1 \leftarrow R$

10. At what point on the graph of $f(x) = \ln(x)$ is the tangent line parallel to the line $x + 5y = 3$?

$f'(x) = \frac{1}{x}$

slope of tangent line

$5y = -x + 3$

$y = -\frac{1}{5}x + \frac{3}{5}$

parallel ⇒ same slope

$\frac{1}{x} = -\frac{1}{5}$

$x = -5$

$y = \ln(-5)$ DNE

11. A bacteria culture doubles every 6 hours. How long will it take to triple in size?

$y = Pe^{rt}$

$2P = Pe^{r \cdot 6}$

$2 = e^{6r}$

$\ln(2) = 6r$

$r = \frac{\ln(2)}{6}$

$3P = Pe^{\frac{\ln(2)}{6}t}$

$3 = e^{\frac{\ln(2)}{6}t}$

$\ln(3) = \frac{\ln(2)}{6}t$

$\boxed{\frac{6\ln(3)}{\ln(2)} = t}$ hours

12. Use logarithmic differentiation to find the derivative of each of the following.

(a) $y = (3x + 1)^{\tan(x)}$

$\ln(y) = \ln(3x+1)^{\tan(x)}$

$\ln(y) = \tan(x) \cdot \ln(3x+1)$

differentiate ↓ product rule

$\frac{1}{y} \cdot y' = \left[\tan(x) \cdot \frac{3}{3x+1} + \ln(3x+1) \cdot \sec^2(x)\right]$

$\boxed{y' = \left[\tan(x) \cdot \frac{3}{3x+1} + \ln(3x+1) \cdot \sec^2(x)\right] \cdot (3x+1)^{\tan(x)}}$

(b) $y = (\ln(x))^{x^4-7}$

$\frac{1}{x\ln(x)}$

$\ln(y) = \ln\left((\ln(x))^{x^4-7}\right) = (x^4-7)\cdot \ln(\ln(x))$

Differentiate $\frac{1}{y}\cdot y' = (x^4-7)\cdot \frac{1}{\ln(x)}\cdot \frac{1}{x} + \ln(\ln(x))\cdot(4x^3)$

$$\boxed{y' = \left[(x^4-7)\cdot\frac{1}{\ln(x)}\cdot\frac{1}{x} + \ln(\ln(x))\cdot 4x^3\right]\cdot(\ln(x))^{x^4-7}}$$

13. Find the quadratic with equation $y = ax^2 + bx$ whose tangent line at $x = 2$ has equation $y = 4x + 6$.

slope

Slope $m = 4$

$y' = 2ax + b$

$y'(2) = 4a + b$

$4 = 4a + b$

Point (quadratic) $y(2) = 4a + 2b$

tangent line $y(2) = 14$

$14 = 4a + 2b$

$4 = 4a + b$
$14 = 4a + 2b$

$(4 = 4a + b)(-1)$
$14 = 4a + 2b$

$-4 = -4a - b$
$14 = 4a + 2b$

$\boxed{10 = b}$

$4 = 4a + 10$
$4a = -6$
$\boxed{a = -\frac{3}{2}}$

$y = -\frac{3}{2}x^2 + 10x$

14. If $f(x) = \ln(\sin^2(x))$, find $f'(\frac{\pi}{2})$.

(a) $f'(\frac{\pi}{2}) = 2$
(b) $f'(\frac{\pi}{2}) = -1$
(c) $f'(\frac{\pi}{2}) = \frac{1}{2}$
(d) $f'(\frac{\pi}{2}) = 0$
(e) $f'(\frac{\pi}{2}) = 2\sqrt{3}$

$\ln((\sin(x))^2) = 2\ln(\sin(x))$

$f'(x) = \frac{1}{\sin^2 x}\cdot 2(\sin(x))\cdot\cos(x)$

$= \frac{2\sin x\cos x}{\sin^2 x} = \frac{2\cos x}{\sin(x)}$

$\boxed{(\sin(x))^2}$

$2(\sin(x))'\cdot\cos(x)$

15. Find $\frac{dy}{dx}$ for $\sin(xy^3) - \tan(4x) = 2x^3 + 3^{y^2}$.

$3^{y^2}$

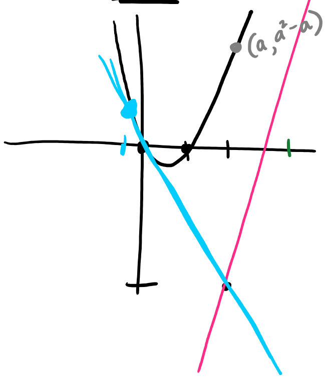
$\cos(xy^3)\left[x\cdot 3y^2\frac{dy}{dx} + y^3\right] - \sec^2(4x)\cdot 4 = 6x^2 + 3^{y^2}\cdot 2y\cdot\ln(3)\frac{dy}{dx}$

$\cos(xy^3)\,x3y^2\frac{dy}{dx} + \cos(xy^3)\,y^3 - 4\sec^2(4x) = 6x^2 + 3^{y^2}\cdot 2y\cdot\ln(3)\frac{dy}{dx}$

$\cos(xy^3)\,y^3 - 4\sec^2(4x) - 6x^2 = \left(3^{y^2}2y\cdot\ln(3)\frac{dy}{dx} - \cos(xy^3)\,x3y^2\frac{dy}{dx}\right)$

$$\boxed{\frac{\cos(xy^3)y^3 - 4\sec^2(4x) - 6x^2}{3^{y^2}\cdot 2y\ln(3) - \cos(xy^3)x3y^2} = \frac{dy}{dx}}$$

16. Find the equations of the lines through the point $\boxed{(2,-7)}$ that are tangent to the parabola $y = x^2 - x$.




Point(s):

$(2,-7)$

$\boxed{(a, a^2-a)}$

Slope

$y' = 2x - 1$

@ $x = a$

$\boxed{m = 2a-1}$

Equation of tangent line:

$(a^2-a)-(-7) = (2a-1)(a-2)$

$a^2 - a + 7 = 2a^2 - 4a - a + 2$

$0 = a^2 - 4a - 5$

$0 = (a-5)(a+1) \Rightarrow a = -1, 5$

$y - y_1 = m(x - x_1)$

$y + 7 = m(x-2)$

$\boxed{\begin{array}{l} ① \; y+7 = -3(x-2) \\ ② \; y+7 = 9(x-2) \end{array}}$

17. Find the derivative of $y = \arccos(e^{3x})$.

(a) $\cancel{f'(x) = \frac{3e^{3x}}{\sqrt{1+e^{6x}}}}$

(b) $\cancel{f'(x) = \frac{3e^{3x}}{\sqrt{1-e^{6x}}}}$

(c) $f'(x) = -\frac{3e^{3x}}{\sqrt{1-e^{6x}}}$

(d) $\cancel{f'(x) = -\frac{3}{\sqrt{1-e^{6x}}}}$

(e) $f'(x) = -\frac{3e^{3x}}{1+e^{6x}}$

$(\arccos(x))' = \frac{-1}{\sqrt{1-x^2}}$

$(e^{3x})^2 = e^{3x \cdot 2} = e^{6x}$

$\frac{-1}{\sqrt{1-(e^{3x})^2}} \cdot 3e^{3x}$

18. Find $f''(0)$ if $f(x) = (5-x^2)^3$.

$f'(x) = 3(5-x^2)^2 \cdot (-2x)$

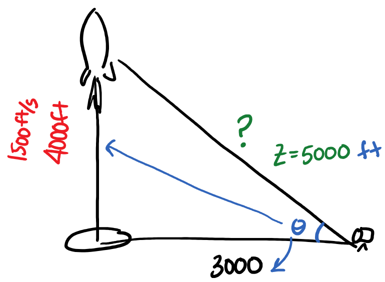
$= -6x(5-x^2)^2$

$f''(x) = -6x[2(5-x^2)^1(-2x)] + (5-x^2)^2 \cdot (-6)$

$f''(0) = 0 - 150 = \boxed{-150}$

19. A camera is positioned 3000 feet from the base of a rocket launching pad. At a particular moment, the rocket rises vertically. Its speed is 1500 ft/s when it has risen 4000 ft.

(a) How fast is the distance from the camera to the rocket changing at that moment?




$$x^2+y^2=z^2$$

$$(3000)^2+y^2=z^2$$

differentiate

$$2y\cdot\frac{dy}{dt}=2z\cdot\frac{dz}{dt}$$

$$2\cdot 4000\cdot 1500=2(5000)\frac{dz}{dt}$$

$$12{,}000{,}000=10{,}000\frac{dz}{dt}$$

$$\boxed{\frac{dz}{dt}=1200\ ft/s}$$

(b) If the camera is focused on the rocket, how fast is the camera's angle of elevation changing at that moment?

$$\tan(\theta)=\frac{y}{3000}=\frac{1}{3000}y$$

differentiate

$$\sec^2(\theta)\cdot\frac{d\theta}{dt}=\frac{1}{3000}\cdot\frac{dy}{dt}$$

$$\left(\frac{5000}{3000}\right)^2\frac{d\theta}{dt}=\frac{1}{3000}\cdot 1500$$

$$\frac{25}{9}\frac{d\theta}{dt}=\frac{1}{2}$$

$$\boxed{\frac{d\theta}{dt}=\frac{9}{50}\ \frac{rad}{sec}}$$

20. $f(x) = \begin{cases} ax^2 - 2bx + 8 & \text{if } x \leq 2 \\ bx - 1 & \text{if } x > 2 \end{cases}$

(a) What must be true for $f(x)$ to be continuous everywhere?

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (ax^2 - 2bx + 8) = 4a - 4b + 8$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (bx - 1) = 2b - 1$$

$$\boxed{4a - 4b + 8 = 2b - 1}$$

(b) Find the values of $a$ and $b$ that make $f(x)$ differentiable everywhere, if possible. If not possible, explain why.

$$f'(x) = \begin{cases} 2ax - 2b & \text{if } x \leq 2 \\ b & \text{if } x > 2 \end{cases}$$

$$\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^-} (2ax - 2b) = 4a - 2b$$

$$\lim_{x\to 2^+} f'(x) = \lim_{x\to 2^+} (b) = b$$

$$\boxed{4a - 2b = b}$$

$$4a - 6b + 9 = 0$$
$$(4a - 3b = 0)(-1)$$
$$-4a + 3b = 0$$

$$-3b + 9 = 0$$
$$9 = 3b$$
$$\boxed{b = 3}$$

$$4a - 3(3) = 0$$
$$\boxed{a = \tfrac{9}{4}}$$

21. Find the derivative of $f(x) = \ln\left(\frac{e^{3x}(2x+7)^4}{\sqrt[3]{x^2 - 5}}\right)$

Rewrite $= \ln(e^{3x}) + \ln(2x+7)^4 - \ln(\sqrt[3]{x^2-5})$

$= 3x + 4\ln(2x+7) - \frac{1}{3}\ln(x^2 - 5)$

$$\boxed{f'(x) = 3 + 4\frac{1}{2x+7}\cdot 2 - \frac{1}{3}\frac{1}{x^2-5}\cdot 2x}$$