Wir 5: Sections 14.5, 14.6

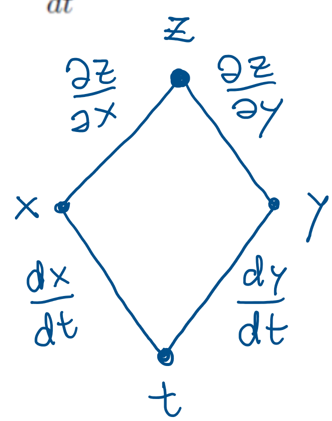
Section 14.5

Problem 1. If $z = \ln(9x - 6y)$, $x = \cos(e^t)$, $y = \sin^3(4t)$, find $\dfrac{dz}{dt}$.

$$z_x = \frac{9}{9x-6y} \qquad z_y = \frac{-6}{9x-6y}$$

$$\frac{dx}{dt} = -e^t \sin(e^t)$$

$$\frac{dy}{dt} = 4 \cdot 3 \sin^2(4t)\cos(4t)$$

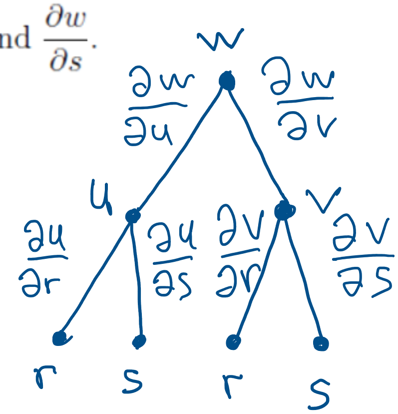


$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} =$$

$$= \frac{-9e^t \sin(e^t)}{9x-6y} - \frac{6 \cdot 12 \sin^2(4t)\cos(4t)}{9x-6y} =$$

$$= \frac{-9e^t \sin(e^t)}{+9\cos(e^t) - 6\sin^3(4t)} - \frac{72 \sin^2(4t)\cos(4t)}{9\cos(e^t) - 6\sin^3(4t)} \quad \#$$

Problem 2. If $w = u^2 + 2uv$, $u = r \ln s$, $v = 2r + s$, find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$.



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial r} =$$

$$= (2u+2v)\ln s + (0+2u)(2) =$$

$$= 2\ln s(r\ln s + 2r + s) + 4r\ln s =$$

$$= 2\ln s(r\ln s + 4r + s)$$

$$\frac{\partial w}{\partial s} = w_u u_s + w_v v_s =$$

$$= (2u+2v)\frac{r}{s} + (0+2u)(1) = 2(r\ln s + 2r + s)\frac{r}{s} + 2r\ln s =$$

$$= 2\left(\frac{r^2}{s}\ln s + \frac{2r^2}{s} + r + r\ln s\right)$$

$$= 2\left(\frac{r^2}{s}\ln s + \frac{2r^2}{s} + r + r\ln s\right)$$

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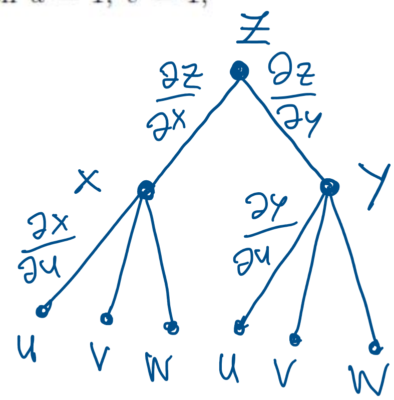
Problem 3. If $z = x^4 + xy^3$, $x = uv^3 + w^4$, $y = u + ve^w$, find find $\dfrac{\partial z}{\partial u}$ when $u = 1$, $v = 1$, $w = 0$.



$$Z_u = Z_x X_u + Z_y Y_u =$$

$$= (4x^3 + y^3)v^3 + (0 + 3xy^2)\cdot 1$$

$$(4 + 8)\cdot 1 + 3\cdot 1\cdot 4\cdot 1 =$$

$$= 12 + 12 = 24$$

when $u = v = 1$ and $w = 0$,
$x = 1 + 0 = 1$ and $y = 1 + 1e^0 = 2$
$x = 1$, $y = 2$

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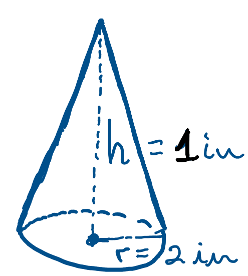
Problem 4. The height and radius of a right circular cone are changing with respect to time. If the base radius of the cone is increasing at a rate of $\frac{1}{4}$ inches per minute while the height is decreasing at a rate $\frac{1}{10}$ inches per minute, find the rate in which the volume of the cone is changing when the radius of the cone is 2 inches and the height of the cone is 1 inch.

$$\frac{dr}{dt} = \frac{1}{4}\ \frac{in}{min}\ ; \quad \frac{dh}{dt} = \frac{-1}{10}\ \frac{in}{min}$$



$$Vol = \frac{1}{3}\pi r^2 \cdot h$$

$$\frac{dV}{dt} = V_r\, r_t + V_h\, h_t =$$

$$\frac{2\pi}{3} rh \cdot \frac{1}{4} + \frac{\pi}{3} r^2\left(-\frac{1}{10}\right) = \frac{\cancel{2}\pi}{3}\cdot\cancel{2}\cdot\frac{1}{\cancel{4}} - \frac{\pi}{\cancel{30}_{15}}\cdot\cancel{4}^{2} =$$

$$= \frac{2\pi}{3} rh \cdot \frac{1}{4} + \frac{\pi}{3} r^2 \left(-\frac{1}{10}\right) = \frac{\cancel{2}\pi}{3} \cdot \cancel{2} \cdot \frac{1}{\cancel{4}} - \frac{\pi}{\cancel{30}_{15}} \cdot \cancel{4}^{2} =$$

$$= \frac{\pi}{3} - \frac{2\pi}{15} = \frac{5\pi - 2\pi}{15} = \frac{3\pi}{15} = \frac{\pi}{5} \frac{cm^3}{min}$$

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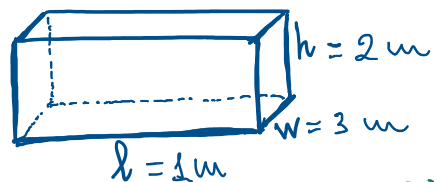
Problem 5. The length l, width w and height h of a box change with time. At a certain instant, the dimensions are $l = 1$ m, $w = 3$ m and $h = 2$ m, and l and w are increasing at rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that same instant, find the rate at which the surface area is changing.

$\frac{dl}{dt} = \frac{dw}{dt} = 2 \frac{m}{s}$; $\frac{dh}{dt} = -3 \frac{m}{s}$



$\frac{dS}{dt} = ?$

$S = 2(lw + wh + hl)$

$$\frac{dS}{dt} = S_l l_t + S_w w_t + S_h h_t =$$

$$= 2(w+h) \cdot 2 + 2(l+h) \cdot 2 + 2(w+l) \cdot (-3) =$$

$$= 2\left[5 \cdot 2 + 3 \cdot 2 - 4 \cdot 3\right] = 2(10 + 6 - 12) = 2 \cdot 4 = 8 \frac{m^2}{s}$$

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Section 14.6

Problem 6. $f(x, y) = xy \sin x$, find the directional derivative at the point $\left(\frac{\pi}{2}, -1\right)$ in the direction $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

Note: $|\vec{u}| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$

$$D_{\vec{u}} f\left(\frac{\pi}{2}, -1\right) = \vec{\nabla} f\left(\frac{\pi}{2}, -1\right) \cdot \vec{u}$$

$$\vec{\nabla} f = \langle y \sin x + xy \cos x, x \sin x \rangle$$

$$\vec{\nabla} f\left(\frac{\pi}{2}, -1\right) = \langle -1 - \cancel{\frac{\pi}{2} \cdot 0}, \frac{\pi}{2} \cdot 1 \rangle = \langle -1, \frac{\pi}{2} \rangle$$

$$\vec{\nabla} f\left(\tfrac{\pi}{2}, -1\right) = \langle -1 - \cancel{\tfrac{\pi}{2} \cdot 0}, \tfrac{\pi}{2} \cdot 1 \rangle = \langle -1, \tfrac{\pi}{2} \rangle$$

$$D_{\vec{u}} f\left(\tfrac{\pi}{2}, -1\right) = \langle -1, \tfrac{\pi}{2} \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle = -\frac{3}{5} + \frac{4\pi}{10} =$$

$$= \frac{-6+4\pi}{10} = \frac{2\pi - 3}{5}.$$

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Problem 7. Given $f(x, y) = x^3y^2$, find the directional derivative at the point $(-1, 2)$ in the direction $4\mathbf{i} - 3\mathbf{j}$. $= \vec{v}$

Note: $|\vec{v}| = \sqrt{16+9} = 5$, $\underline{\underline{\text{not } 1}}$

so, $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{5}(4\hat{\imath} - 3\hat{\jmath}) = \langle \tfrac{4}{5}, -\tfrac{3}{5} \rangle$

$$\vec{\nabla} f = \langle 3x^2y^2, 2x^3y \rangle$$

$$\vec{\nabla} f(-1, 2) = \langle 12, -4 \rangle$$

$$D_{\vec{u}} f(-1, 2) = \langle 12, -4 \rangle \cdot \langle \tfrac{4}{5}, -\tfrac{3}{5} \rangle =$$

$$= \frac{48}{5} + \frac{12}{5} = \frac{60}{5} = 12$$

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Problem 8. If $f(x, y) = x^2e^{xy}$, find the rate of change of f at the point $\boxed{(1, 0)}$ (P_0) in the direction of the point $P(1, 0)$ to the point $Q(5, 2)$.

$P_0 = P$

$$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u}$$

$$\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{1}{|\vec{PQ}|}(Q - P) =$$

$$\vec{\nabla} f = \langle 2xe^{xy} + y \cdot x^2 ye^{xy}, x^2 \cdot x e^{xy} \rangle$$

$$\frac{1}{\ } \langle 4, 2 \rangle =$$

$$\vec{\nabla} f = \langle 2xe^{xy} + x^2 y e^{xy}, x^2 \cdot x e^{xy} \rangle$$

$$\vec{\nabla} f(1,0) = \langle 2\cdot 1 + 0, 1\cdot 1 \rangle = \langle 2, 1 \rangle$$

$$D_{\vec{u}} f(1,0) = \langle 2,1 \rangle \cdot \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle =$$

$$= \frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

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$$= \frac{1}{\sqrt{16+4}} \langle 4, 2 \rangle =$$

$$= \langle \frac{4}{\sqrt{20}}, \frac{2}{\sqrt{20}} \rangle =$$

$$= \langle \frac{4}{2\sqrt{5}}, \frac{2}{2\sqrt{5}} \rangle = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle = \vec{u}$$

Problem 9. Find the gradient of $f(x, y) = x^2 + y^3 - 4xy$ at the point $(1, -1)$.

$$\vec{\nabla} f = \langle f_x, f_y \rangle = \langle 2x + 0 - 4y, 0 + 3y^2 - 4x \rangle$$

$$\vec{\nabla} f(1,-1) = \langle 2+4, 3-4 \rangle = \langle 6, -1 \rangle$$

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Problem 10. If $f(x, y) = x^2 e^{-2y}$, $P(2, 0)$, $Q(-3, 1)$.

a.) Find the directional derivative at Q in the direction of P.

b.) Find a vector in the direction in which f increases most rapidly at P, and find the rate of change of f in that direction.

$$D_{\vec{u}} f(Q) = \vec{\nabla} f(Q) \cdot \vec{u}$$

$$\vec{\nabla} f = \langle 2x e^{-2y}, -2x^2 e^{-2y} \rangle$$

$$\vec{u} = \frac{\vec{QP}}{|\vec{QP}|} = \frac{1}{|\vec{QP}|} (P - Q) =$$

$$= \frac{1}{\sqrt{\ldots}} \langle 5, -1 \rangle =$$

$$\vec{\nabla} f = \langle 2x e^{-2y}, -2x^2 e^{-2y} \rangle$$

$$\vec{\nabla} f(-3,1) = \langle -6 e^{-2}, -18 e^{-2} \rangle =$$

$$= -6 e^{-2} \langle 1, 3 \rangle$$

$$= \frac{1}{\sqrt{25+1}} \langle 5, -1 \rangle =$$

$$= \langle \frac{5}{\sqrt{26}}, -\frac{1}{\sqrt{26}} \rangle = \vec{u}$$

$$D_{\vec{u}} f(-3,1) = -\frac{6}{e^2} \langle 1, 3 \rangle \cdot \langle \frac{5}{\sqrt{26}}, \frac{-1}{\sqrt{26}} \rangle =$$

$$= -\frac{6}{e^2} \left(\frac{5}{\sqrt{26}} - \frac{3}{\sqrt{26}} \right) = -\frac{6}{e^2} \cdot \frac{2}{\sqrt{26}} = \frac{-12}{e^2 \sqrt{26}}$$

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Problem 11. Find the maximum rate of change of $f(x, y) = \sin^2(3x + 2y)$ at the point $\underbrace{\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)}_{P_0}$ and the direction in which it occurs.

The max rate of change of f at P_0 equals $|\vec{\nabla} f(P_0)|$, and it occurs in the direction of $\vec{\nabla} f(P_0)$.

$$\vec{\nabla} f = \langle 3 \cdot 2 \sin(3x+2y) \cos(3x+2y), 2 \cdot 2 \sin(3x+2y) \cos(3x+2y) \rangle$$

$$3\frac{\pi}{6} - 2\frac{\pi}{8} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{2\pi - \pi}{4} = \frac{\pi}{4}$$

$$\vec{\nabla} f\left(\frac{\pi}{6}, -\frac{\pi}{8}\right) = \langle 6 \sin\frac{\pi}{4} \cos\frac{\pi}{4}, 4 \sin\frac{\pi}{4} \cos\frac{\pi}{4} \rangle =$$

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{2}{4} = \frac{1}{2}$$

$$= \langle 6 \cdot \frac{1}{2}, \frac{4}{2} \rangle = \langle 3, 2 \rangle$$

* Max rate of change $= \sqrt{9+4} = \sqrt{13}$, in direction of vector $\langle 3, 2 \rangle$.

Problem 12. Find the equation of the tangent plane to the surface $f(x, y) = x^2 + y^2 - 4xy$ at the point $(1, 2)$.

$$z - z_0 = f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0)$$

$$z_0 = f(P_0) = 1 + 4 - 8 = -3$$

$z_0 = f(P_0) = 1 + 4 - 8 = -3$

$f_x = 2x - 4y$ $\qquad$ $f_x(P_0) = 2 - 8 = -6$

$f_y = 2y - 4x$ $\qquad$ $f_y(P_0) = 4 - 4 = 0$

$Z + 3 = -6(x-1) + 0(y-2)$ $\qquad$ $+6 \;-3$

$$\boxed{-6x - Z + 3 = 0}$$