

NOTE: ALL 15.3 PROBLEMS WERE SOLVED LAST WEEK

### Section 15.3

Recall: If  $P(x, y)$  is a point in the  $xy$ -plane, we can represent the point  $P$  in polar form: Let  $r$  be the distance from  $O$  to  $P$  and let  $\theta$  be the angle between the polar axis and the line  $OP$ . Then the point  $P$  is represented by the ordered pair  $(r, \theta)$ , and  $r, \theta$  are called the **polar coordinates** of  $P$ .

Connecting polar coordinates with rectangular coordinates:

a.)  $x = r \cos(\theta), y = r \sin(\theta)$

b.)  $\tan(\theta) = \frac{y}{x}$ , thus  $\theta = \arctan\left(\frac{y}{x}\right)$ .

c.)  $x^2 + y^2 = r^2$

**Problem 1.** Find the cartesian coordinates of the polar point  $\left(2, \frac{2\pi}{3}\right)$ .

**Problem 2.** Find the polar coordinates of the rectangular point  $(\sqrt{3}, -1)$ .

**Problem 3.** Find a cartesian equation for the curve described by  $r = 2 \sin \theta$ .

**Problem 4.** Find a polar equation for  $y = 1 + 3x$

**Change to Polar Coordinates in a Double Integral:** If  $f$  is continuous on a polar rectangle  $R$  given by  $0 \leq a \leq r \leq b$ ,  $\alpha \leq \theta \leq \beta$ , where  $0 \leq \beta - \alpha \leq 2\pi$ , then

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

**Problem 5.** Evaluate  $\iint_R (x + 2) dA$ , where  $R$  is the region bounded by the circle  $x^2 + y^2 = 4$ .

**Problem 6.** Evaluate  $\iint_R 4y dA$ , where  $R$  is the region in the second quadrant bounded by the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .

**Problem 7.** Evaluate  $\iint_R 3x^2 dA$ , where  $R$  is the region in the first quadrant enclosed by the circle  $x^2 + y^2 = 9$  and the lines  $y = 0$  and  $y = x$ .

**Problem 8.** Change  $\int_0^3 \int_0^{\sqrt{9-x^2}} x^2 dy dx$  to a polar double integral. Do not evaluate.

**Problem 9.** Change  $\int_0^4 \int_0^{\sqrt{4x-x^2}} \sqrt{x^2 + y^2} dy dx$  to a polar double integral. Do not evaluate.

**Problem 10.** Set up but do not evaluate a double integral that gives the volume of the solid that lies above the  $xy$ -plane, below the sphere  $x^2 + y^2 + z^2 = 81$  and inside the cylinder  $x^2 + y^2 = 4$

**Problem 11.** Find the volume of the solid bounded by the paraboloids  $z = 20 - x^2 - y^2$  and  $z = 4x^2 + 4y^2$ .

Wir 8: Sections 15.6, 15.7, 15.8

### Section 15.6

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, we now define triple integrals for functions of three variables.

Definition: The **Triple Integral** of  $f$  over the box  $E = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$  is

$$\iiint_E f(x, y, z) dV = \iiint_E f(x, y, z) dx dy dz$$

1. Evaluate  $\iiint_E xyz^2 dV$  where  $E = [0, 1] \times [-1, 2] \times [0, 3]$

1. Evaluate  $\iiint_E xyz^2 dV$  where  $E = [0, 1] \times [-1, 2] \times [0, 3]$

$$\int_0^1 \int_{-1}^2 \int_0^3 xyz^2 dz dy dx$$

$$\left. \frac{27}{2} \frac{x^2}{2} \right|_0^1 = \frac{27}{4} (1-0) = \frac{27}{4}$$

$$xz^2 \frac{1}{2} y^2 \Big|_{y=-1}^{y=2} = \frac{1}{2} xz^2 (4-1) = \frac{3}{2} xz^2$$

$$\frac{1}{2} x \frac{1}{3} z^3 \Big|_{z=0}^{z=3} = \frac{1}{2} x (27-0) = \frac{27}{2} x$$

2. Evaluate  $\int_0^1 \int_x^{x^2} \int_x^y xyz dz dy dx$

$$4 \times 10$$

$$4 \times 4$$

$$4 \times 6$$

$$8 \cdot 3 \cdot 5$$

$$\frac{2^4}{5}$$

$$\frac{1}{120}$$

$$xy \frac{z^2}{2} \Big|_{z=x}^{z=y} = \frac{1}{2} xy (y^2 - x^2) = \frac{1}{2} (xy^3 - x^3y)$$

$$\frac{1}{2} \left[ x \frac{1}{4} y^4 - x^3 \frac{1}{2} y^2 \right]_{y=x}^{y=x^2} = \frac{1}{2} \left[ \frac{1}{4} x^9 - \frac{1}{2} x^7 - \frac{1}{4} x^5 + \frac{1}{2} x^5 \right]$$

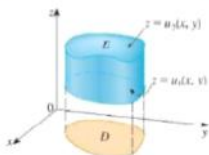
$$= \frac{1}{2} \left[ \frac{1}{4} \frac{x^{10}}{10} - \frac{1}{2} \frac{x^8}{8} + \frac{1}{4} \frac{x^6}{6} \right]_0^1 = \frac{1}{2} \left( \frac{1}{40} - \frac{1}{16} + \frac{1}{24} \right) = \frac{1}{2} \left( \frac{3-7.5+5}{120} \right) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{120} = \frac{1}{480}$$

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### Triple Integrals over a general bounded region $E$ in three dimensional space:

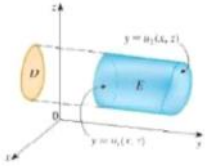
**Type I:** A solid region  $E$  is said to be of type I if it lies between the graphs of two continuous functions of  $x$  and  $y$ , that is  $E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$  where  $D$  is the projection of  $E$  on the  $xy$ -plane. Notice that the upper bound of  $E$  is the surface  $z = u_2(x, y)$  and the lower bound of  $E$  is the surface  $z = u_1(x, y)$ . Moreover, it can be shown that

$$\iiint_E f(x, y, z) dV = \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA$$



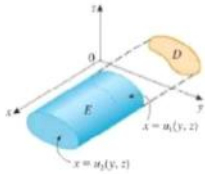
**Type II:** A solid region  $E$  is said to be of type II if it lies between the graphs of two continuous functions of  $x$  and  $z$ , that is  $E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$  where  $D$  is the projection of  $E$  on the  $xz$ -plane. Notice that the right bound of  $E$  is the surface  $y = u_2(x, z)$  and the left bound of  $E$  is the surface  $y = u_1(x, z)$ . Moreover, it can be shown that

$$\iiint_E f(x, y, z) dV = \iint_D \left[ \int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z) dy \right] dA$$



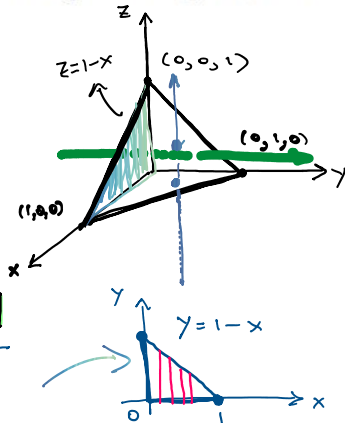
**Type III:** A solid region  $E$  is said to be of type III if it lies between the graphs of two continuous functions of  $y$  and  $z$ , that is  $E = \{(x, y, z) | (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$  where  $D$  is the projection of  $E$  on the  $yz$ -plane. Notice that the back surface of  $E$  is  $x = u_1(y, z)$  and the front surface of  $E$  is  $x = u_2(y, z)$ . Moreover, it can be shown that  $\iiint_E f(x, y, z) dV =$

$$\iint_D \left[ \int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) dx \right] dA$$



3. Evaluate  $\iiint_E z dV$  where  $E$  is the solid tetrahedron bounded by the four planes  $x = 0, y = 0, z = 0$  and  $x + y + z = 1$ .

$$\begin{cases} 0 \leq z \leq 1-x-y \\ 0 \leq y \leq 1-x \\ 0 \leq x \leq 1 \end{cases}$$



$$\int_0^1 \int_0^{1-x} \left\{ \int_0^{1-x-y} z dz \right\} dy dx$$

$$\frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2 dy dx$$

$$\frac{1}{2} \left[ (1-x-y)^2 - x^2 + xy - y + xy + y^2 \right]_{y=0}^{y=1-x}$$

$$\frac{1}{2} \left[ \frac{1}{3} (-1)(1-x-y)^3 \right]_{y=0}^{y=1-x} = -\frac{1}{6} \left[ 0 - (1-x)^3 \right] dx$$

$$+\frac{1}{6} (-1) \frac{1}{4} (1-x)^4 \Big|_0^1 = -\frac{1}{24} (0^4 - 1^4) = \frac{1}{24}$$

$$\begin{cases} 0 \leq y \leq 1-x-z \\ 0 \leq z \leq 1-x \\ 0 \leq x \leq 1 \end{cases}$$

4. Evaluate  $\iiint_E x dV$  where  $E$  is the solid bounded by the four planes  $x = 0, y = 0, z = 0$  and  $4x + 2y + z = 6$ .

$$\begin{cases} 0 \leq z \leq 6 - 4x - 2y \\ \uparrow (0,0,6) \end{cases}$$

4. Evaluate  $\iiint_E x \, dV$  where  $E$  is the solid bounded by the four planes  $x = 0$ ,  $y = 0$ ,  $z = 0$  and

$$4x + 2y + z = 6.$$

$$\int_0^{3/2} \int_0^{3-2x} \int_0^{6-4x-2y} x \, dz \, dy \, dx$$

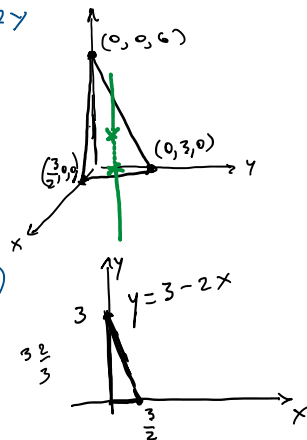
$$\left[ (6x - 4x^2)y - xy^2 \right]_{y=0}^{y=3-2x} = 6x - 4x^2 - 2xy$$

$$\int_0^{3/2} (6x - 4x^2)(3-2x) - x(3-2x)^2 \, dx$$

$$= \int_0^{3/2} (18x - 12x^2 - 12x^2 + 8x^3 - 9x + 12x^2 - 4x^3) \, dx$$

$$= \int_0^{3/2} (4x^3 - 12x^2 + 9x) \, dx$$

$$= \left[ x^4 - 4x^3 + \frac{9}{2}x^2 \right]_0^{3/2} = \frac{81}{16} - 4 \frac{27}{8} + \frac{9}{2} \cdot \frac{9}{4} = \frac{81 - 216 + 162}{16} = \frac{27}{16}$$



5. Evaluate  $\iiint_E xz \, dV$  where  $E$  is the solid tetrahedron with vertices points  $(0, 0, 0)$ ,  $(0, 1, 0)$ ,  $(1, 1, 0)$  and  $(0, 1, 1)$ .

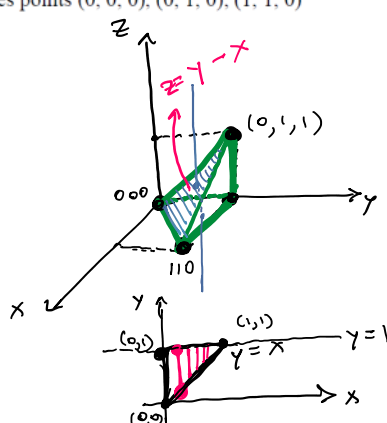
$$\int_0^1 \int_x^1 \int_0^{y-x} xz \, dz \, dy \, dx$$

$$\left[ \frac{1}{2} xz^2 \right]_{z=0}^{z=y-x} = \frac{1}{2} x(y-x)^2$$

$$\int_x^1 \frac{1}{2} x(y-x)^2 \, dy = \frac{1}{6} x \left[ (1-x)^3 - 0^3 \right]$$

$$= \frac{1}{6} x (1 - 3x + 3x^2 - x^3)$$

$$\int_0^1 (x - 3x^2 + 3x^3 - x^4) \, dx = \left[ \frac{1}{2}x^2 - x^3 + \frac{3}{4}x^4 - \frac{x^5}{5} \right]_0^1 = \frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} = \frac{1}{120}$$

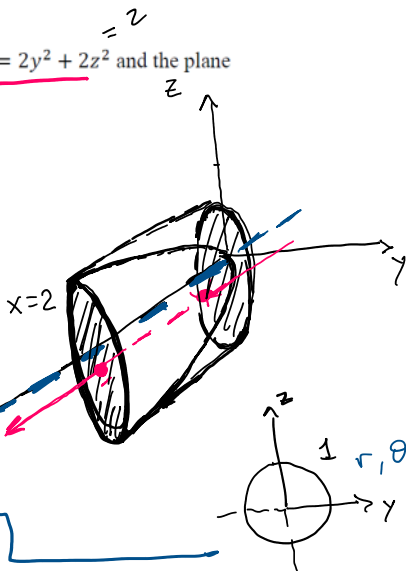


6. Evaluate  $\iiint_E x \, dV$  where  $E$  is the 3D region bounded by the paraboloid  $x = 2y^2 + 2z^2$  and the plane  $x = 2$ .

$$\int_0^{2\pi} \int_0^1 \int_{2r^2}^2 x \, dx \, r \, d\theta$$

$$\left[ \frac{1}{2} x^2 \right]_{2r^2}^2 = \frac{1}{2} (4 - 4r^4)$$

$$\int_0^1 (2 - 2r^4) \, r \, dr = \left[ r^2 - \frac{2}{5} r^5 \right]_0^1 = 1 - \frac{2}{5} = \frac{3}{5}$$



Answer:  $\frac{4\pi}{3}$

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{1-x} \sqrt{x^2+z^2} dz dy dx \cdot 2\pi$$

Answer:  $\frac{128}{15} \pi$

7. Evaluate  $\iiint_E \sqrt{x^2+z^2} dV$  where  $E$  is the region bounded by the paraboloid  $y = x^2 + z^2$  and the plane  $y = 4$ .

$$\int_0^{2\pi} \int_0^2 \int_{r^2}^4 r dy r dr d\theta$$

$$x^2 + z^2 \leq y \leq 4$$

$$r^2 \leq y \leq 4$$

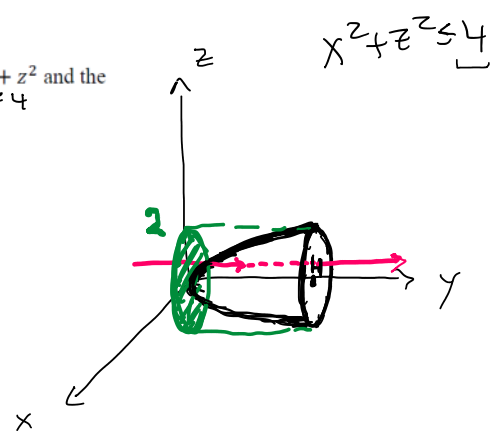
$$0 \leq r \leq 2$$

$$0 \leq \theta \leq 2\pi$$

$$(4-r^2)r^2 = 4r^2 - r^4$$

$$\left[ \frac{4}{3}r^3 - \frac{r^5}{5} \right]_0^2 = \frac{4}{3} \cdot 8 - \frac{32}{5} - 0 = \frac{32}{3} - \frac{32}{5}$$

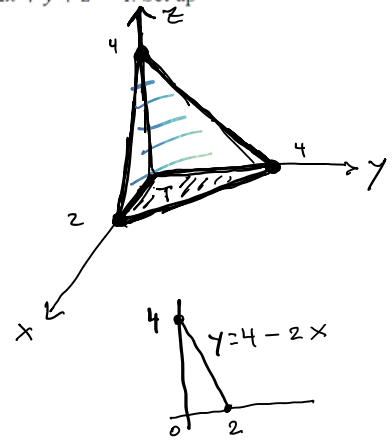
$$\frac{160 - 96}{15} = \frac{64}{15} \cdot 2\pi \quad \text{Answer } \frac{128}{15} \pi$$



Note: We can use a triple integral to find the volume of a solid  $E$  because  $Vol(E) = \iiint_E 1 dV = \iint_R [\text{top} - \text{bottom}] dA$

8. Consider the tetrahedron enclosed by the three coordinate planes and the plane  $2x + y + z = 4$ . Set up but do not evaluate:

- a) a double integral that gives the volume of this solid;
- b) a triple integral that gives the volume of this solid.



a)  $\iint_T [(4-2x-y) - 0] dA = \int_0^2 \int_0^{4-2x} (4-2x-y) dy dx$

b)  $\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz dy dx$

9. Find the volume of the solid bounded by the cylinder  $x = y^2$  and the planes  $z = 0$  and  $x + z = 1$ .

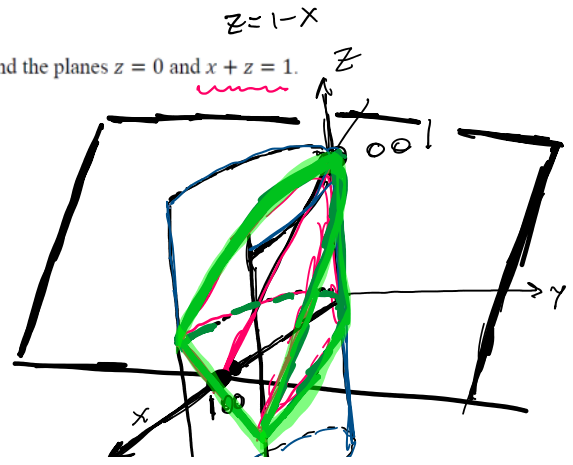
$$0 \leq z \leq 1-x$$

$$0 \leq x \leq 1$$

$$-\sqrt{x} \leq y \leq \sqrt{x}$$

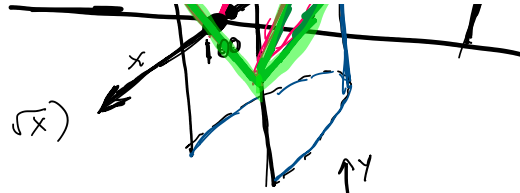
$$Vol = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{1-x} 1 dz dy dx$$

$$\int_0^1 (1-x) dy$$



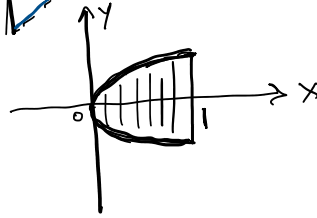


$$\int_{y=-\sqrt{x}}^{y=\sqrt{x}} (1-x) dy = (1-x)(\sqrt{x} - (-\sqrt{x}))$$



$$\int 2\sqrt{x} - x^{3/2}$$

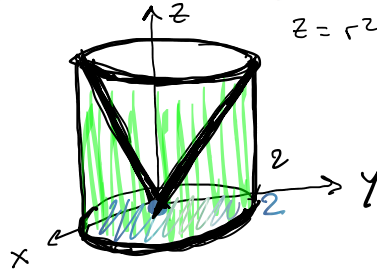
$$2 \cdot \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} \Big|_0^1 = \frac{4}{3} - \frac{2}{5} = \frac{20-6}{15} = \frac{14}{15}$$



Section 15.7

10. Use cylindrical coordinates to calculate the volume above the  $xy$ -plane outside the cone  $z = x^2 + y^2$  and inside the cylinder  $x^2 + y^2 = 4$ .

$$Vol = \int_0^{2\pi} \int_0^2 [r^2 - 0] r dr d\theta$$



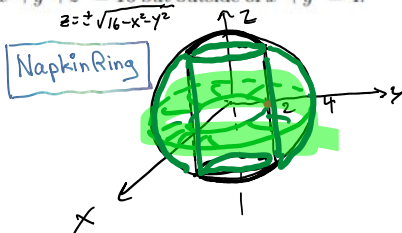
$$\left[ \frac{r^4}{4} \right]_0^2 = \frac{16}{4} - 0 = 4 \cdot 2\pi = 8\pi$$

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11. Consider the surfaces  $x^2 + y^2 + z^2 = 16$  and  $x^2 + y^2 = 4$ . Set up a triple integral in cylindrical coordinates which can be used to calculate the volume of the solid which is inside of  $x^2 + y^2 + z^2 = 16$  but outside of  $x^2 + y^2 = 4$ .

Calculate the volume.

$$Vol = \int_0^{2\pi} \int_2^4 \int_{-\sqrt{16-r^2}}^{+\sqrt{16-r^2}} 1 dz r dr d\theta$$



$$u = 16 - r^2$$

$$du = -2r dr$$

$$= \int_0^{2\pi} \int_2^4 2\sqrt{16-r^2} \cdot r dr d\theta$$

$$= \int_0^{2\pi} \int_0^{12} u^{1/2} du d\theta$$

$$= + \int_0^{2\pi} u^{3/2} du = \frac{2}{3} u^{3/2} \Big|_0^{12} = \frac{2}{3} [12^{3/2} - 0] \cdot 2\pi$$

$$Vol = \frac{4}{3} \pi \cdot 12^{3/2}$$

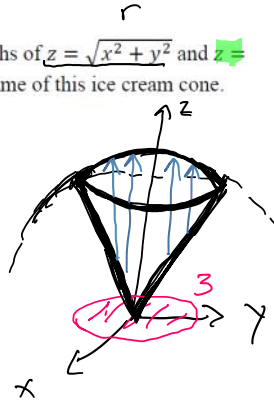
12. Consider the solid shaped like an ice cream cone that is bounded by the graphs of  $z = \sqrt{x^2 + y^2}$  and  $z = r$

12. Consider the solid shaped like an ice cream cone that is bounded by the graphs of  $z = \sqrt{x^2 + y^2}$  and  $z = \sqrt{18 - x^2 - y^2}$ . Set up an integral in cylindrical coordinates to find the volume of this ice cream cone.

$$r = \sqrt{18 - r^2} \rightarrow r^2 = 18 - r^2$$

$$2r^2 = 18 \rightarrow r^2 = 9$$

$$Vol = \int_0^{2\pi} \int_0^3 \int_r^{\sqrt{18-r^2}} 1 \, dz \, r \, dr \, d\theta$$



$$\int_0^3 (r\sqrt{18-r^2} - r^2) \, dr = \dots$$

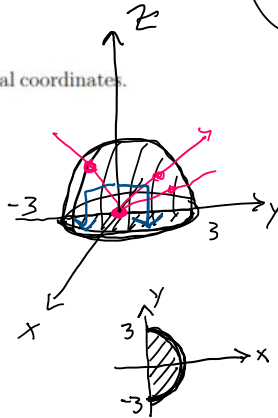
$$\rho^2 \sin^2 \varphi = \rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta$$

$\uparrow \quad \quad \quad \uparrow$   
 $x^2 + y^2 \quad \quad \quad z^2$

13. Consider the integral  $\int_{-3}^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} (x^2 + y^2) \, dz \, dx \, dy$ .

Convert the given integral from rectangular coordinates to cylindrical coordinates.

$$\int_{-\pi/2}^{\pi/2} \int_0^3 \int_0^{\sqrt{9-r^2}} r^2 \, dz \, r \, dr \, d\theta$$

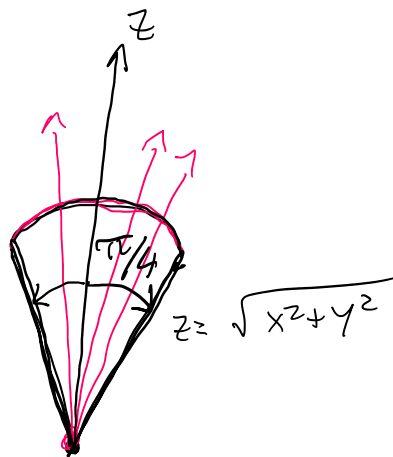


14. Solve problem #12 with spherical coordinates.

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \rho \leq \sqrt{18}$$

$$0 \leq \varphi \leq \frac{\pi}{4}$$



$$Vol = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{18}} 1 \, \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

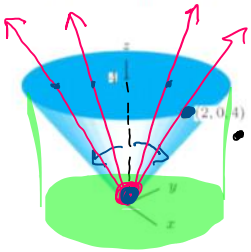
$$\text{Vol} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

15. Convert the integral in problem #13 to an equivalent one in spherical coordinates.

$$\begin{cases} -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ 0 \leq \rho \leq 3 \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{cases} \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 (\rho^2 \sin^2 \varphi) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

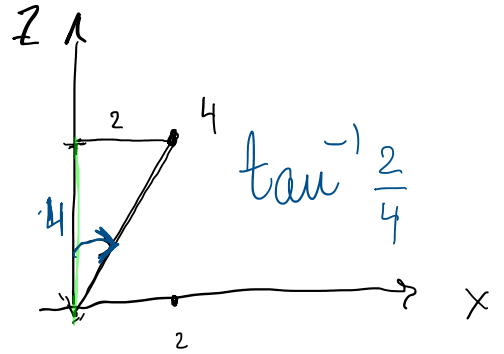
$$\int \sin^3 \quad \int \sin^4 \left[ \frac{1}{2} (1 - \cos 2\theta) \right]^2$$

16. Set up the volume of the region sketched below in spherical coordinates.



$$4 = \rho \cos \varphi \implies \rho = \frac{4}{\cos \varphi} = 4 \sec \varphi$$

$$\cos \varphi = \frac{z}{\rho} \implies \rho \cos \varphi = z = 4$$



$$\begin{aligned} 0 &\leq \theta \leq 2\pi \\ 0 &\leq \varphi \leq \tan^{-1}\left(\frac{1}{2}\right) \\ 0 &\leq \rho \leq 4 \sec \varphi \end{aligned}$$

$$\text{Vol} = \int_0^{2\pi} \int_0^{\tan^{-1} \frac{1}{2}} \int_0^{4 \sec \varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

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