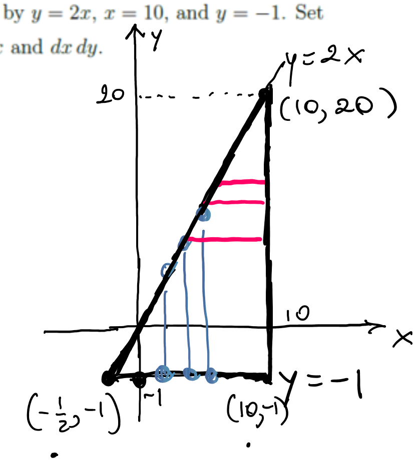
Wir 9: Exam 3 Review

Sections 15.1-15.~~4~~ 3, 15.6-15.~~9~~ 8

Problem 1. Let R be the region in the xy–plane bounded by $y = 2x$, $x = 10$, and $y = -1$. Set up but do not evaluate $\iint_R (x^2 + y^2)\,dA$ in the order $dy\,dx$ and $dx\,dy$.



$\begin{cases} -\frac{1}{2} \le x \le 10 \\ -1 \le y \le 2x \end{cases}$ $\begin{cases} -1 \le y \le 20 \\ \frac{y}{2} \le x \le 10 \end{cases}$

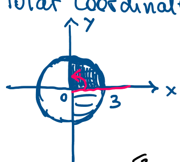
$$\int_{-\frac{1}{2}}^{10}\int_{-1}^{2x} (x^2+y^2)\,dy\,dx = \int_{-1}^{20}\int_{\frac{y}{2}}^{10} (x^2+y^2)\,dx\,dy$$

#

Problem 2. Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} e^{-x^2-y^2}\,dy\,dx$ Polar Coordinates

$e^{-(x^2+y^2)}$

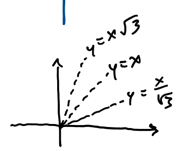
$\begin{cases} 0 \le r \le 3 \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$



$$\int_0^{\pi/2}\int_0^3 e^{-r^2} r\,dr\,d\theta$$

$u = -r^2$
$du = -2r\,dr$

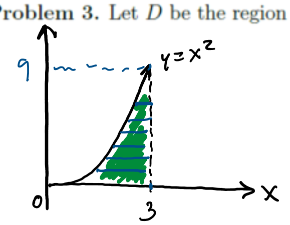
$$\int_0^{-9} -\frac{1}{2}e^u\,du = \int_{-9}^0 \frac{1}{2}e^u\,du = \frac{1}{2}e^u\Big|_{-9}^0 =$$



Answer $= \frac{1}{2}(1 - e^{-9})\frac{\pi}{2} = \frac{\pi}{4}\left(1 - \frac{1}{e^9}\right)$

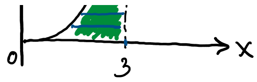
#

Problem 3. Let D be the region bounded by $y = 0$, $y = x^2$, and $x = 3$. Find $\iint_D 3x \cos y\,dA$.



$$\int_0^3\int_0^{x^2} 3x\,\underbrace{\cos y\,dy}\,dx$$

$$\sin y\Big|_0^{x^2} = \sin(x^2) - 0$$



0 $\quad \sin y\Big|_0^{x^2} = \sin(x^2) - 0$

$u = x^2$
$du = 2x\,dx$

$$\int_0^3 3x(\sin x^2)\,dx = \int_0^9 3\frac{1}{2}\sin u\,du =$$

$$= \frac{3}{2}\Big[-\cos u\Big]_0^9 = \frac{3}{2}\Big[-\cos 9 - -1\Big] =$$

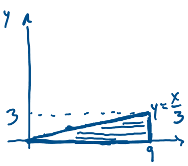
$$= \frac{3}{2}\left(1 - \cos 9\right)$$

#

$$\int_0^9 \int_{\sqrt{y}}^3 3x\cos y\,dx\,dy$$

$$\frac{3}{2}x^2\Big|_{\sqrt{y}}^3 = \int_0^9 \frac{3}{2}(9-y)\cos y\,dy \quad \ldots$$

Problem 4. Compute $\int_0^3 \int_{3y}^9 7e^{x^2}\,dx\,dy$.



$x = 3y$
$y = \frac{x}{3}$

$$\int_0^9 \int_0^{x/3} 7e^{x^2}\,dy\,dx$$

$$\int_0^9 7\frac{x}{3}e^{x^2}\,dx$$

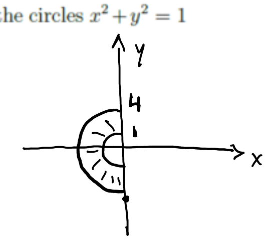
$$\frac{7}{3}\frac{1}{2}e^{x^2}\Big|_0^9 = \frac{7}{6}\left(e^{81} - 1\right)$$

#

Problem 5. Let R be the region that lies to the left of the y-axis between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$. Find $\int\int_R 5(x + y)\,dA$.

Polar

$$\begin{cases} 1 \le r \le 4 \\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \end{cases}$$



$$5\int_1^4 \int_{\pi/2}^{3\pi/2} (r\cos\theta + r\sin\theta)\,r\,d\theta\,dr$$

$-1 \quad -1$

$$\Big[r^2\sin\theta - r^2\cos\theta\Big]_{\pi/2}^{3\pi/2} = r^2\Big((-1-0) - (1-0)\Big) = \qquad \frac{16}{4}$$

$$= -2r^2$$

$$-5\int_1^4 2r^2\,dr = -10\frac{1}{3}r^3\Big|_1^4 = -\frac{10}{3}\left(4^3 - 1\right) = -\frac{10}{3}(64 - 1) = -\frac{10}{\cancel{3}}\cancel{63}^{21} =$$

$$-5\int_1^2 r^2\,dr = -10\frac{1}{3} r^3\Big|_1 = -\frac{10}{3}(4^3-1) = -\frac{10}{3}(64-1) = -\frac{10}{\cancel{3}}\,\overset{21}{\cancel{63}} =$$

$$= -210$$

#

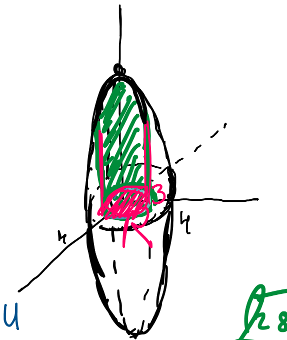
Problem 6. Find the volume of the sold that is above the xy plane, below the ellipsoid $4x^2+4y^2+z^2=64$ but inside the cylinder $x^2+y^2=9$.

$4(x^2+y^2) = 64 \to z = +\sqrt{64-4x^2-4y^2}$

$x^2+y^2 = 16$

$$\int_0^{\sqrt{64-r^2}}$$

$$Vol = \iint_R \text{Top} - \overset{=0}{\text{Bottom}}\, dA$$



$$\int_0^{2\pi}\int_0^3 \sqrt{64-4r^2}\; r\,dr\,d\theta$$

$64-4r^2 = u$

$-8r\,dr = du$

$64-36$

$\boxed{28}$

$$-\int_{64}^{28} u^{1/2}\frac{du}{8} = \int_{28}^{64}\frac{1}{8}u^{1/2}\,du = \frac{1}{\underset{4}{\cancel{8}}}\,\frac{\overset{1}{\cancel{2}}}{3}\, u^{3/2}\Big|_{28}^{64} = \frac{1}{12}\left(\underset{8^3}{64^{3/2}} - 28^{3/2}\right)$$

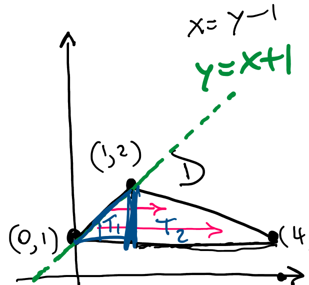
ANSWER $\frac{\pi}{6}\left(8^3 - 28^{3/2}\right).$ #

$28^{3/2} \to 8\sqrt{7}^3$ $\quad 56\sqrt{7}$

Problem 7. Let D be the triangular region with vertices $(0,1)$, $(1,2)$, and $(4,1)$. Set up but do not evaluate $\int\int_D 7y^2\,dA$ in the order $dy\,dx$ and $dx\,dy$.

$$\begin{cases} 1 \le y \le 2 \\ y-1 \le x \le 7-3y \end{cases}$$

$$T_1 \begin{cases} 0 \le x \le 1 \\ 1 \le y \le x+1 \end{cases} \qquad T_2 \begin{cases} 1 \le x \le 4 \\ 1 \le y \le \frac{7}{3} - \frac{1}{3}x \end{cases}$$



$$\int_1^2\int_{y-1}^{7-3y} 7y^2\,dx\,dy$$

$$\int_0^1\int_1^{x+1} 7y^2\,dy\,dx + \int_1^4\int_1^{7/3-\frac{1}{3}x} 7y^2\,dy\,dx$$

#

$y = -\frac{1}{3}x + \frac{7}{3}$

$\cancel{3}x = \frac{7}{\cancel{3}} - y3$

$x = -3y + 7$

Problem 8. Let $D = \{(x,y) : 0 \le x \le 1, 0 \le y \le x^2\}$. Evaluate

$$\int\int_D \frac{5y}{6x^5+1}\,dA. = \int_0^1\int_0^{x^2} \frac{5y}{6x^5+1}\,dy\,dx =$$

$$\frac{1}{2}y^2\Big|_0^{x^2} =$$

$$= \frac{1}{2}(x^4 - 0)$$

$$\int_0^1 \frac{1}{2}\frac{5\;\; x^4}{6x^5+1}\,dx$$

$$\int_0^1 \frac{1}{2}\,\frac{\boxed{5\ x^4}}{6\boxed{x^5}+1}\,dx \qquad\qquad \frac{1}{2}y^2\Big|_0 = \frac{1}{2}(x^4-0)$$

$$u = 6x^5+1 \qquad du = 30x^4\,dx$$

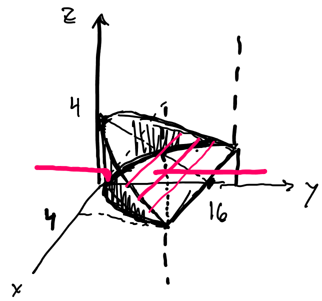
$$\int_1^7 \frac{1}{2}\,\frac{1}{u}\,\frac{du}{\cancel{30}\,6} =$$

$$= \frac{1}{12}\ln u\Big|_1^7 = \frac{1}{12}\left(\ln 7 - \cancel{\ln 1}\right) = \frac{1}{12}\ln 7 = \ln 7^{1/12}.$$

#

Problem 9. Express $\iiint_E f(x,y,z)\,dV$ in the order $dydzdx$ if E is the solid bounded by $y = x^2$, $z = 0$, $y + 4z = 16$.

$$\int_{-4}^{4}\int_{0}^{4-\frac{1}{4}x^2}\int_{x^2}^{16-4z} f(x,y,z)\,dy\,dz\,dx$$

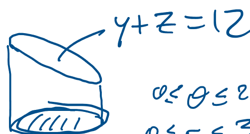


$$0 \le z \le \frac{16}{4} - \frac{y}{4} = 4 - \frac{y}{4}$$

#

Problem 10. Find the volume of the solid that is enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 12$ and $z = 2$.

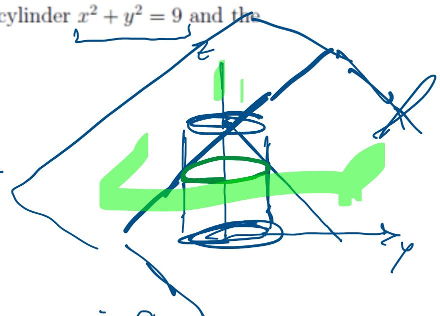
$z = 12 - y$



$z=2$

$0 \le \theta \le 2\pi$

$0 \le r \le 3$



$$Vol = \iint_R \int_2^{12-y} 1\,dz\,dA = \int_0^{2\pi}\int_0^3\int_2^{12-r\sin\theta} 1\,dz\,r\,dr\,d\theta =$$

$\iint_R \int_2 \qquad \int_0 \int_0 \int_2 \qquad =$

$45\theta + 9\cos\theta \Big|_0^{2\pi} =$

$45 \cdot 2\pi$

$\boxed{90\pi}$

$12 - r\sin\theta - 2 = 10r - r^2\sin\theta$

$\int 10r - r^2\sin\theta \, dr$

$5r^2 - \frac{r^3}{3}\sin\theta \Big|_0^3$

$\int \underbrace{45 - 9\sin\theta}\, d\theta$

$9\cos 2\pi - 9\cos 0$

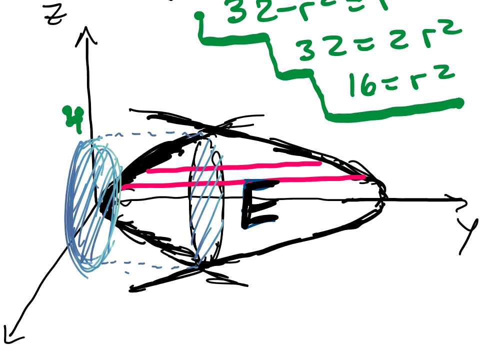
$9 - 9$

Problem 11. Find the volume of the solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 32 - x^2 - z^2$.

$y = r^2$

$= 32 - r^2$

$\underbrace{x^2+z^2} \le y \le \underbrace{32 - x^2 - z^2}$



$32 - r^2 = r^2$

$32 = 2r^2$

$16 = r^2$

$\text{Vol} = \iiint 1 \, dV =$

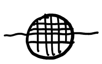
$= \int_0^{2\pi}\int_0^4\int_{r^2}^{32-r^2} 1 \, dy \, r \, dr \, d\theta$

$32 - r^2 - r^2 = 32 - 2r^2$

$\int 32r - 2r^3 \, dr = 16r^2 - \frac{2}{4}r^4 \Big|_0^4 = 16 \cdot 16 - \frac{1}{2}4^4 - 0$

$\frac{1}{2} \cdot 16 \cdot 16 = 128 * 2\pi \quad \boxed{256\pi} = \text{vol}(E)$

Problem 12. Convert to Cylindrical: $\int_{-9}^{9}\int_{-\sqrt{81-y^2}}^{\sqrt{81-y^2}}\int_{\sqrt{x^2+y^2}=r}^{13} xz \, dz \, dx \, dy.$

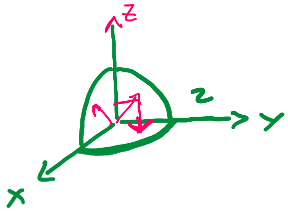


$\int_0^{2\pi}\int_0^{9}\int_r^{13} r\cos\theta \, z \, dz \, r \, dr \, d\theta$

#

Problem 13. Find $\iiint_E (x^2+y^2+z^2)\,dV$ where E is the part of the ball centered at the origin with radius 2 in the first octant.

ρ^2



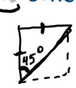
$$\int_0^{\pi/2}\int_0^{\frac{\pi}{2}}\int_0^{2} \rho^2\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta =$$

$$\frac{1}{5}\rho^5\Big|_0^2 = \frac{32}{5}$$

$$\left(\frac{\pi}{2}\right)\frac{32}{5}\left[-\cos\varphi\right]_0^{\pi/2} = \frac{16\pi}{5}\left[-0 - -1\right] = \boxed{\frac{16}{5}\pi}$$

Problem 14. Evaluate in spherical coordinates. $\int_0^{10}\int_0^{\sqrt{100-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{\boxed{200}-x^2-y^2}} yz\,dz\,dy\,dx$ sphere, cone 45°

$z = r$



$200 - r^2 = r^2$

$200 = 2r^2$

$r^2 = 100$

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{200}} \rho\sin\varphi\sin\theta\; \rho\cos\varphi\; \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$\frac{1}{5}\rho^5\Big|_0^{\sqrt{200}} = \; = \frac{1}{5}\, 200 * 200\sqrt{200}$$

$$\boxed{\frac{1}{5}\, 200^{5/2}}$$

$$\int \sin^2\varphi\cos\varphi = \frac{1}{3}\sin^3\varphi\Big|_0^{\pi/4} =$$

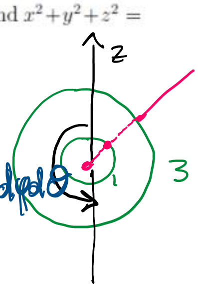
$$= \frac{1}{3}\left[\left(\frac{\sqrt{2}}{2}\right)^3 - 0\right] = \boxed{\frac{1}{3}\,\frac{\cancel{2}\sqrt{2}}{\cancel{8}_4}}$$

$$-\cos\theta\Big|_0^{\pi/2} = -0 - -1 = \boxed{1}$$

$$\frac{1}{12}\,\frac{\sqrt{2}}{5}\,200^{5/2}$$

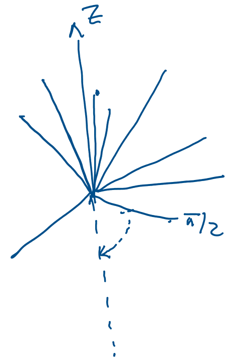
Problem 15. Let E be the region that lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$. Set up but do not evaluate $\iiint_E (x+y+z)\,dV$ in spherical coordinates.

$$\int_0^{2\pi}\int_0^{\pi}\int_1^{3} (\rho\sin\varphi\cos\theta + \rho\sin\varphi\sin\theta + \rho\cos\varphi)\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$



#

z



Problem 16. Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy plane and below the cone $z=\sqrt{x^2+y^2}$.

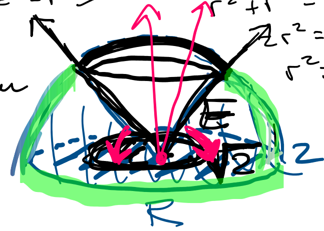
$r^2+z^2=4$

$z=r$ $\quad z^2=r^2$

$r^2+r^2=4$

$2r^2=4$

$r^2=2$



$\mathrm{Vol}(E)=\frac{1}{2}$ Vol. sphere $-$ Vol icecream

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} 1\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$\frac{\rho^3}{3}\Big|_0^2=\frac{8}{3}\quad\left[-\cos\varphi\right]_0^{\frac{\pi}{4}}=\left(-\frac{\sqrt2}{2}--1\right)\quad 2\pi$$

$$2\pi\frac{8}{3}\left(1-\frac{\sqrt2}{2}\right)=\frac{16}{3}\pi\left(1-\frac{\sqrt2}{2}\right)$$

Answer $\frac{1}{2}\,\frac{4}{3}\pi\,8-\frac{16}{3}\pi\left(1-\frac{\sqrt2}{2}\right)$

OR... replace middle integral with

$$\ldots\int_{\pi/4}^{\pi/2}\ldots$$

Problem 17. Let R be the triangular region with vertices $(0,0)$, $(9,1)$, $(1,9)$. Using the transformation $x=9u+v$ and $y=u+9v$ find $\int\int_R(x-10y)dA$.

Problem 18. Let R be the parallelogram enclosed by the lines $x-6y=0$, $x-6y=9$, $6x-y=7$, $6x-y=10$. Using the transformation $u=x-6y$ and $v=6x-y$, find $\int\int_R 9\frac{x-6y}{6x-y}dA$.

Problem 19. Let R be the region bounded by $25x^2 + 4y^2 = 100$. Using the transformation $x = 2u$ and $y = 5v$, find $\int\int_R 4x^2 dA$.