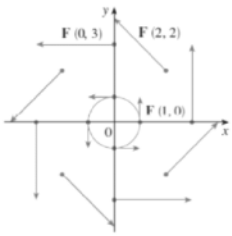
WIR 10

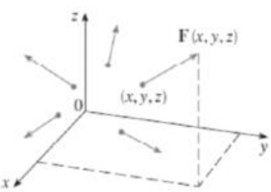
Wir 10: Sections 16.1, 16.2, 16.3, 16.4, 16.5

Section 16.1

Definition: A **vector field** in two dimension is a function $\mathbf{F}$ that assigns to each point (x, y) in $D \subset \mathbb{R}^2$ a two dimensional vector, $\mathbf{F}(x, y)$.
In two dimension, the vector field lies entirely in the xy plane.
Here is a vector field in $\mathbb{R}^2$:



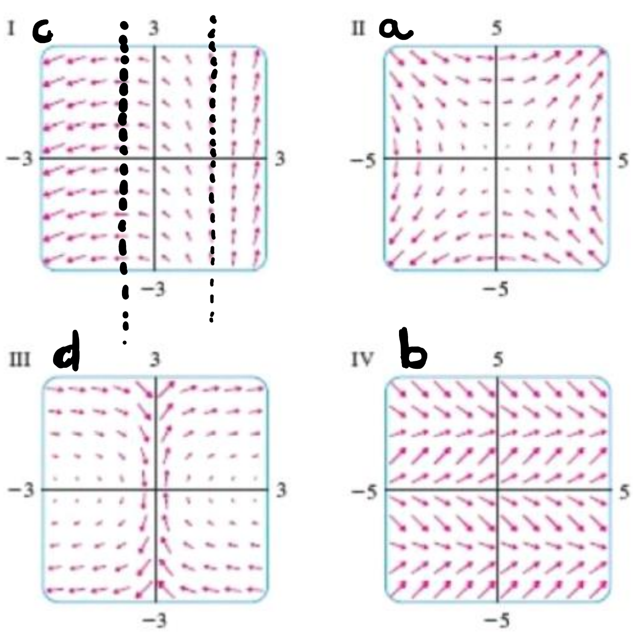
Definition: A **vector field** in three dimension is a function $\mathbf{F}$ that assigns to each point (x, y, z) in $D \subset \mathbb{R}^3$ a three dimensional vector, $\mathbf{F}(x, y, z)$.
In three dimension, the vector field is in space.
Here is a vector field in $\mathbb{R}^3$:



In order to match $\mathbf{F}$ with its vector field, choose a several points, (x, y), in each quadrant, and look at the *direction* of $\mathbf{F}(x, y)$. To narrow down further, look at the behavior of the components. Often times, it is a process of elimination.

Problem 1. Match each vector field equation with its graph:

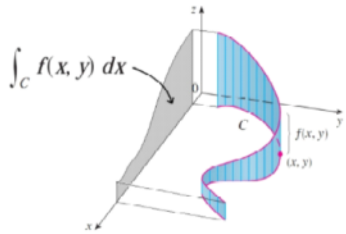
a) $\boldsymbol{F}(x,y) = < y, x >$
b) $\boldsymbol{F}(x,y) = < 1, \sin y >$
c) $\boldsymbol{F}(x,y) = < x-2, x+1 >$
d) $\boldsymbol{F}(x,y) = < y, \frac{1}{x} >$



Section 16.2

Definition: If f is defined on a smooth curve C defined as $\mathbf{r}(t)=\langle x(t),y(t)\rangle$, $a\le t\le b$, then the **line integral of f along C** is

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,}_{ds}dt = \int_a^b (f(x(t),y(t))|\mathbf{r}'(t)|\,dt$$



In order to find a line integral along a curce C, we must first parameteterize the curve. Sometimes, the parameterization will be given explicitly, other times you must parameterize the curve.

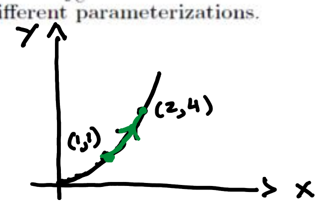
Problem 2. Evaluate $\int_C (2x+y)ds$, where C is defined as $\mathbf{r}(t)=\langle \overset{x}{2+t}, \overset{y}{3-t}\rangle$, $0\le t\le 1$.

$$ds = \sqrt{1^2+(-1)^2}\,dt = \sqrt{2}\,dt$$

$$\int_0^1 (4+2t+3-t)\sqrt{2}\,dt = \sqrt{2}\int_0^1 (7+t)\,dt =$$

$$= \sqrt{2}\left[7t+\frac{1}{2}t^2\right]_0^1 = \sqrt{2}\left(7+\frac{1}{2}\right) = \frac{15\sqrt{2}}{2}$$

Problem 3. Set up but do not evaluate $\int_C (2x+x^2y)ds$, where C is the arc of the curve $y=x^2$ from $(1,1)$ to $(2,4)$ using two different parameterizations.



$\vec{r}_1(t) = \langle \overset{x}{t}, \overset{y}{t^2} \rangle$

$1 \le t \le 2$

$$ds = \sqrt{(1)^2+(2t)^2}\,dt = \sqrt{1+4t^2}\,dt$$

$$\int_1^2 (2t + t^2\cdot t^2)\sqrt{1+4t^2}\,dt$$

$e^t = 2$

$t = \ln 2$

$\vec{r}_2(t) = \langle e^t, e^{2t} \rangle$

$0 \le t \le \ln 2$

$$\int_0^{\ln 2} (2e^t + e^{2t}e^{2t})\sqrt{e^{2t}+4e^{4t}}\,dt$$

$$\int_0 (2e^{t}+e^{-t}e)\sqrt{e+4e}\,dt$$

$$ds=\sqrt{(e^t)^2+(2e^{2t})^2}\,dt=$$
$$=\sqrt{e^{2t}+4e^{4t}}\,dt$$

Problem 4. Evaluate $\int_C (x^2+y)\,ds$ where C consists of the line segment from the point $(1,4)$ A to $(3,-1)$. B

$$A + t\overrightarrow{AB}$$

$$\vec r(t)=\langle 1,4\rangle + t\underbrace{\langle 2,-5\rangle}$$

$$\vec r(t)=\langle 1+2t,\ 4-5t\rangle \qquad 0\le t\le 1$$

$$ds=\sqrt{2^2+(-5)^2}\,dt=\sqrt{29}\,dt$$

$$\int_0^1\left[(1+2t)^2+(4-5t)\right]\sqrt{29}\,dt=$$

$$1+4t+4t^2+4-5t=$$
$$=4t^2-t+5$$

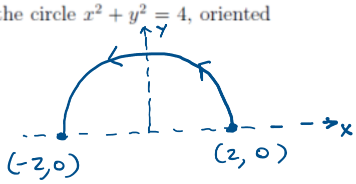
$$=\int_0^1\sqrt{29}\,(4t^2-t+5)\,dt=$$

$$=\sqrt{29}\left[\frac{4}{3}-\frac{1}{2}+5\right]=\sqrt{29}\,\frac{8-3+30}{6}=$$

$$=\frac{35}{6}\sqrt{29}$$

Problem 5. Evaluate $\int_C (x+y)ds$, where C is the top half of the circle $x^2+y^2=4$, oriented counterclockwise.

$$0\le t\le \pi$$



$$\vec r(t)=\langle 2\cos t,\ 2\sin t\rangle$$

$$ds=\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt=2\,dt$$

$$\int_0^\pi 2(\cos t+\sin t)\,2\,dt=$$

$$=4\left[\sin t-\cos t\right]_0^\pi=-4(-1-1)=8.$$

$$= 4\left[\sin t - \cos t\right]_0^{\pi} = -4(-1-1) = 8.$$

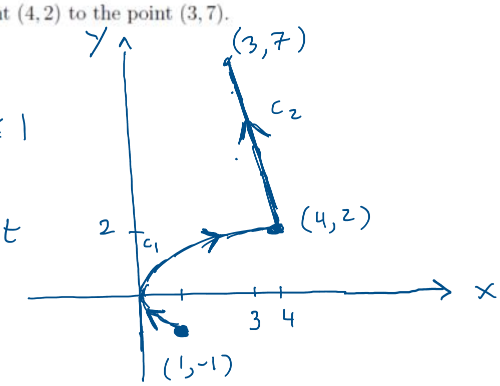
Problem 6. Set up but do not evaluate $\int_C (2 + x^2 y)ds$, where C is the arc of the curve $x = y^2$ from $(1, -1)$ to $(4, 2)$ and then along the line segment from the point $(4, 2)$ to the point $(3, 7)$.

$C_1: \vec{r}_1(t) = \langle t^2, t \rangle \quad -1 \le t \le 2$

$C_2: r_2(t) = \langle 4 - t, 2 + 5t \rangle \quad 0 \le t \le 1$

$C_1: ds = \sqrt{(2t)^2 + 1^2}\, dt = \sqrt{1 + 4t^2}\, dt$

$C_2: ds = \sqrt{(-1)^2 + 5^2}\, dt = \sqrt{26}\, dt$



$$\int_{-1}^{2} (2 + t^4 t)\sqrt{1+4t^2}\, dt + \int_0^1 \left(2 + (4-t)^2(2+5t)\right)\sqrt{26}\, dt$$

Line Integrals over vector fields: Suppose now are moving a particle along a curve C through a vector (force) field, $\mathbf{F}$. We define the **line integral of F along C** to be

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_a^b \underbrace{(\mathbf{F}(\mathbf{r}(t))}\cdot \mathbf{r}'(t)dt$$

Problem 7. Find $\int_c \vec{F} \cdot d\vec{r}$, where C is defined by $\mathbf{r}(\mathbf{t}) = \langle t, t^2, t^4 \rangle$, $0 \le t \le 1$, and $\mathbf{F}(x, y, z) = \langle x, z^2, -4y \rangle$.

$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \langle t, t^8, -4t^2 \rangle \cdot \langle 1, 2t, 4t^3 \rangle =$
$= t + 2t^9 - 16t^5$

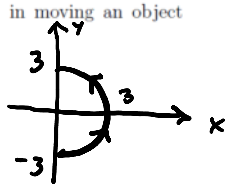
$$= t + 2t^9 - 16t^5$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (t + 2t^9 - 16t^5)\, dt =$$

$$= \frac{1}{2} + 2\frac{1}{10} - 16\frac{1}{6} = \frac{1}{2} + \frac{1}{5} - \frac{8}{3} = \frac{15+6-80}{30} = -\frac{59}{30}$$

Problem 8. Find the work done by the force field $\mathbf{F}(x, y) = \langle x^2, xy\rangle$ in moving an object counterclockwise around the right half of the circle $x^2 + y^2 = 9$.

$$W = \int_C \vec{F} \cdot d\vec{r} \qquad \begin{cases} x = 3\cos t \\ y = 3\sin t \\ -\frac{\pi}{2} \le t \le \frac{\pi}{2} \end{cases}$$



$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) =$$

$$= \langle 9\cos^2 t,\ 9\cos t \sin t\rangle \cdot \langle -3\sin t,\ 3\cos t\rangle =$$

$$= -27\cos^2 t \sin t + 27\cos^2 t \sin t = 0.$$

Answer: 0.

Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ for $a \le t \le b$. The line integral of f along C with respect to x is $\int_C f(x,y)dx = \int_a^b (f(x(t), y(t))\ \underbrace{x'(t)\,dt}_{dx}$.

The line integral of f along C with respect to y is $\int_C f(x,y)dy = \int_a^b (f(x(t), y(t))\ \underbrace{y'(t)\,dt}_{dy}$

Problem 9. $\int_C y\,dx + x^2 dy$, where C is decribed by $\vec{\mathbf{r}}(t) = \langle 3e^t, e^{2t}\rangle$, $0 \le t \le 1$.

dy

Problem 9. $\int_C y\,dx + x^2 dy$, where C is decribed by $\vec{r}(t) = \langle 3e^t, e^{2t}\rangle$, $0 \le t \le 1$.

$dx = 3e^t dt$
$dy = 2e^{2t} dt$

$$\int_0^1 e^{2t} \cdot 3e^{2t} dt + 9e^{2t} \cdot 2e^{2t} dt =$$

$$= \int_0^1 (3e^{4t} + 18e^{4t})\, dt = 21 \int_0^1 e^{4t} dt =$$

$$= 21 \; \frac{1}{4} e^{4t} \Big|_0^1 = \frac{21}{4}(e^4 - 1)$$

Problem 10. Evaluate $\int_C x\,dx + y\,dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.

$\vec{r}(t) = \langle 4 - t^2, t \rangle$

$dx = -2t\,dt \qquad dy = 1\,dt$

$-3 \le t \le 1$

$$\int_{-3}^1 (4 - t^2)(-2t)\,dt + t\,(1)\,dt =$$

$-8t + 2t^3 + t$

$$\int_{-3}^1 (2t^3 - 7t)\,dt = \left[\cancel{2}^1 \frac{t^4}{\cancel{4}_2} - 7\frac{t^2}{2}\right]_{-3}^1 =$$

$$= \frac{1}{2} - \frac{7}{2} - \left(\frac{3^4}{2} - \frac{7}{2} \cdot 9\right) =$$

$$= -\frac{6}{2} - \frac{81}{2} - \frac{63}{2} = -3 - \frac{144}{2} = -3 - 72 = -75$$

63
81
144

Problem 11. Evaluate $\int_C (x+y)dz + (y-x)dy + z\,dx$ where C is decribed by $x = t^4$, $y = t^3$, $z = t^2$, $0 \le t \le 1$.

$dz = 2t\,dt$, $\quad dx = 4t^3 dt$, $dy = 3t^2 dt$

$$\int_0^1 (t^4+t^3)(2t)\,dt + (t^3-t^4)(3t^2)\,dt + (t^2)(4t^3)\,dt =$$

$$(2t^5+2t^4+3t^5-3t^6+4t^5) = -3t^6+9t^5+2t^4$$

$$-\frac{3}{7} + \frac{\cancel{9}^{3}}{\cancel{6}_{2}} + \frac{2}{5} = \frac{-30+105+28}{70} = \frac{103}{70}$$

Section 16.3

In section 16.2, we learned how to find a line integral over a vector field $\mathbf{F}$ along a curve C that is parametrized by $\mathbf{r}(t)$, $a \le t \le b$.

Problem 1. Suppose we are moving a particle from the point $(0,0)$ to the point $(2,4)$ in a force field $\mathbf{F}(x,y) = \langle y^2, x\rangle$. Find $\int_c \mathbf{F}\cdot d\mathbf{r}$,where

a.) The particle travels along the line segment from $(0,0)$ to $(2,4)$.

b.) The particle travels along the curve $y = x^2$ from $(0,0)$ to $(2,4)$.

a) $\vec{r}_1(t) = \langle 2t, 4t\rangle \quad 0 \le t \le 1$

$\vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t) =$

$= \langle 16t^2, 2t\rangle \cdot \langle 2, 4\rangle$

$\int_0^1 (32t^2 + 8t)\,dt =$

$= \frac{32}{3} + \frac{8}{2} = \frac{32}{3} + 4 =$

$= \frac{32+12}{3} = \frac{44}{3}$

b) $\vec{r}_2(t) = \langle t, t^2\rangle \quad 0 \le t \le 2$

$\vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t) =$

$= \langle t^4, t\rangle \cdot \langle 1, 2t\rangle$

$\int_0^2 (t^4 + 2t^2)\,dt =$

$= \left.\frac{t^5}{5} + 2\frac{t^3}{3}\right]_0^2 = \frac{32}{5} + \frac{16}{3} =$

$= \frac{96+80}{15} = \frac{176}{15}$

Note: Although the end points are the same, the value of the line integral is **different** because the **paths** are different. In this section, we will learn under what conditions the line integral is independent of the path taken.

Definition: If $\mathbf{F}$ is a continuous vector field, we say that $\int_c \mathbf{F} \cdot dr$ is **independent of path** if and only if $\int_{C_1} \mathbf{F} \cdot dr = \int_{C_2} \mathbf{F} \cdot dr$ for any two paths C_1 and C_2 with the same starting and ending points. In other words, the line integral is the same **no matter what path** you travel on as long as the endpoints are the same.

Definition: A vector field $\mathbf{F}$ is called a conservative vector field if it is the gradient of some scalar funtion f, that is there exists a function f so that $\mathbf{F} = \nabla f$. We call f the **potential function**.

Problem 2. Consider $f(x, y) = x^2y - y^3$. Find the gradient and explain why it is conservative. What is the potential function?

It is the given f.

$\underbrace{\text{b/c it is a gradient}}$ (definition)

$$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2xy, x^2 - 3y^2 \rangle$$

Recall the Fundamental Theorem of Calculus tells us that $\int_a^b f'(x)dx = f(b) - f(a)$.

Since $\nabla f = \langle f_x, f_y \rangle$, we can think of the potential function, f, as some sort of antiderivative of ∇f. Hence $\int \mathbf{F} \cdot d\mathbf{r} = \int \nabla f \cdot d\mathbf{r}$.

Fundamental Theorem for Line Integrals: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let $\mathbf{F}$ be a conservative vector field. Let f be a differentiable function of two or three variables whose gradient vector, ∇f, is continuous on C. Then

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$$

Note: **Line integrals of conservative vectors fields are independent of path** because in a conservative vector field, the line integral is computed by only using the **endpoints** of the domain! Therefore, if we are in a conservative vector field, the line integral along a curve C in that vector field will be the same **no matter what curve we travel across** that connects the endpoints together. **WHICH MEANS WE DON'T EVEN NEED TO PARAMETERIZE THE CURVE!**

Question: How do we determine if a vector field is conservative, and if so, how do we find the potential function? The 'test for conservative' we use depends on whether $\mathbf{F}$ is in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Theorem: $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = P\mathbf{i} + Q\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.
Note: This above criteria to determine if a vector field is conservative works only for $\mathbb{R}^2$.

Problem 3. Is $\mathbf{F}(x, y) = \langle \overset{P}{3x^2 - 4y}, \overset{Q}{4y^2 - 2x} \rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

$Q_x = -2$, $P_y = -4$

No potential function!

Problem 4. Is $\mathbf{F}(x, y) = \langle \overset{P}{x + y}, \overset{Q}{x - 2} \rangle$ a conservative vector field? If so, find a function f so that $\underline{\mathbf{F} = \nabla f}$.

$Q_x = 1$ $P_y = 1$

conservative!

$\langle f_x , f_y \rangle = \langle x + y , x - 2 \rangle$

$$\langle f_x , f_y \rangle = \langle x+y , x-2 \rangle$$

$$\frac{\partial f}{\partial x} = x+y \Rightarrow f = \frac{x^2}{2} + xy + g(y)$$

$$\frac{\partial f}{\partial y} = x-2 \Rightarrow f = xy - 2y + h(x)$$

$$f(x,y) = \frac{x^2}{2} + xy - 2y + C$$

✓ $\frac{2x}{2} + y$ $\qquad$ $x-2$ ✓

$\frac{16}{3}$ / 48

Problem 5. Given $\mathbf{F}(x, y) = \langle \underbrace{2xy^3}_{P}, \underbrace{3x^2y^2}_{Q} \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by $\mathbf{r}(t) = \langle t^3 + 2t^2 - t, 3t^4 - t^2 \rangle$, $0 \le t \le 2$.

$\vec{r}(2) = \langle 14, 44 \rangle$

$\vec{r}(0) = \langle 0, 0 \rangle$

$Q_x = 6xy^2 \qquad P_y = 6xy^2$

conservative

$$\langle f_x , f_y \rangle = \langle 2xy^3, 3x^2y^2 \rangle$$

$$\frac{\partial f}{\partial x} = 2xy^3 \Rightarrow f = x^2y^3 + g(y)$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 \Rightarrow f = x^2y^3 + h(x)$$

$$f(x,y) = x^2y^3 + C$$

$$\int_C \vec{F} \cdot d\vec{r} \overset{FTLI}{=} \int_C \vec{\nabla} f \cdot d\vec{r} = f(14,44) - f(0,0) =$$

$$= (14^2)(44^3) - 0 .$$

Problem 6. Let $\mathbf{F}(x, y) = \langle \overset{P}{3 + 2xy^2}, \overset{Q}{2x^2y} \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the arc of the

~~parabola y = x^2 from (0, 0) to (1, 1).~~

hyperbola $y = \frac{1}{x}$ from $(1, 1)$ to $\left(4, \frac{1}{4}\right)$.

$\vec{r}(t) = \langle t, \frac{1}{t} \rangle \qquad 1 \le t \le 4$

~~parabola y = x^2 from (0, 0) to (1, 1).~~

hyperbola $y = \frac{1}{x}$ from $(1, 1)$ to $\left(4, \frac{1}{4}\right)$.

$\vec{r}(t) = \langle t, \frac{1}{t} \rangle \qquad 1 \le t \le 4$

$\left.\begin{array}{l} Q_x = 4xy \\ P_y = 4xy \end{array}\right\}$ conserv.

$f_x = 3 + 2xy^2$

$f = 3x + x^2y^2 + g(y)$

$f_y = 2x^2y$

$f = x^2y^2 + h(x)$

$f(x,y) = 3x + x^2y^2 + C$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} =$$

FTLI

$$= \left(3 \cdot 4 + 16 \cdot \frac{1}{16}\right) - (3 + 1) =$$

$$= 8 + 1 = 9.$$

$\qquad\qquad P \qquad Q$

Problem 7. Given $\mathbf{F}(x, y) = \langle 3x^2 - 4y, 4y^2 - 2x \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by $\vec{\mathbf{r}}(t) = \langle t^2, t^2 + t - 2 \rangle$, $0 \le t \le 1$.

$\left.\begin{array}{l} Q_x = -2 \\ P_y = -4 \end{array}\right\}$ not conservative

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) =$$

$$= \langle 3t^4 - 4t^2 - 4t + 8, 4(t^2 + t - 2)^2 - 2t^2 \rangle \cdot \langle 2t, 2t + 1 \rangle =$$

$$= \int_0^1 \text{polynomial } dt \ \ldots$$

Green's Theorem: Let C be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

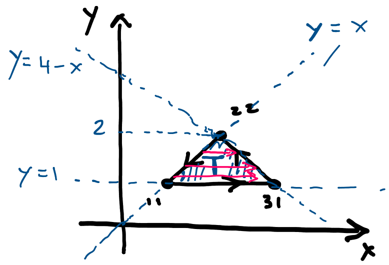
$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

This says that the line integral over a simple closed curve C is equal to a double integral over the area of the region D the curve C encloses.

Note: We only use Green's theorem if we are on a **positively oriented closed** curve. If the curve is not positively oriented, then change the sign of the line integral. If not explicitly stated, assume counterclockwise orientation.

Problem 8. Evaluate $\oint_C \overset{P}{y^2}\, dx + \overset{Q}{x}\, dy$ where C is the triangular path from $(1,1)$ to $(3,1)$ to $(2,2)$ then back to $(1,1)$.

$Q_x = 1 \qquad P_y = 2y$



$$\iint_T (1 - 2y)\, dx\, dy =$$

$$\begin{cases} 1 \le y \le 2 \\ y \le x \le 4 - y \end{cases}$$

$$\int_1^2 \int_y^{4-y} (1 - 2y)\, dx\, dy =$$

$$(1 - 2y)\left[4 - y - y\right] = (1 - 2y)(2 - y)(2) =$$

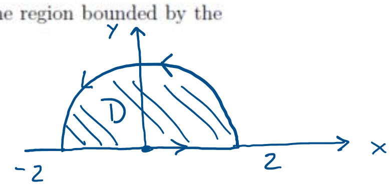
$$= 2(2 - y - 4y + 2y^2) =$$

$$= 2(2 - 5y + 2y^2)$$

$$2\left[2y - \frac{5}{2}y^2 + \frac{2}{3}y^3\right]_1^2 = 2\left[4 - 10 + \frac{16}{3} - 2 + \frac{5}{2} - \frac{2}{3}\right] = 2\left(-8 + \frac{14}{3} + \frac{5}{2}\right) = -\frac{5}{3}$$

Problem 9. Evaluate $\oint_C \overset{P}{y^2}\, dx + \overset{Q}{x^2}\, dy$ where C is the boundary of the region bounded by the semicircle $y = \sqrt{4 - x^2}$ and the x axis. Assume positive orientation.

$Q_x - P_y = 2x - 2y$



$$\iint_D 2(x - y)\, dA =$$

$$\begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le \pi \end{cases}$$

$$= \int_0^\pi \int_0^2 2(r\cos\theta - r\sin\theta)\, r\, dr\, d\theta =$$

$$\underbrace{2r^2}\,\underbrace{(\cos\theta + \sin\theta)}$$

$$\left[\frac{2}{3} r^3\right]^2 \left[\sin\theta - \cos\theta\right]^\pi =$$

$$= \left[\frac{2}{3}r^3\right]_0^2 \left[\sin\theta - \cos\theta\right]_0^{\pi} =$$

$$= \frac{2}{3}\cdot 8(-1)[-1-1] = \frac{32}{3}$$

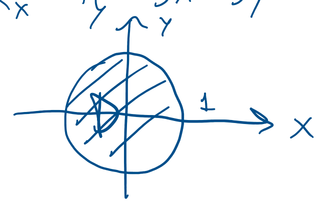
Problem 10. Suppose a particle travels one revolution clockwise around the unit circle under the force field $\mathbf{F}(x, y) = \langle e^x - y^3, \cos(y) + x^3\rangle$. Find the work done.

$$W = \oint_C \vec{F}\cdot d\vec{r} = -\iint_D 3(x^2 - y^2)\,dA = 3\iint_D (y^2 - x^2)\,dA =$$

$P = e^x - y^3$

$Q = (\cos y) + x^3$

$Q_x - P_y = 3x^2 - 3y^2$



$$= \int_0^{2\pi}\int_0^1 \left(r^2 \sin^2\theta - r^2\cos^2\theta\right) r\,dr\,d\theta =$$

$$\tfrac{1}{2}(\cancel{1} - \cos 2\theta) - \tfrac{1}{2}(\cancel{1} + \cos 2\theta)$$

$$= \left[\frac{1}{4}r^4\right]_0^1 \left[-\frac{1}{2}\sin 2\theta\right]_0^{2\pi} =$$

$$-\frac{1}{2}\cos 2\theta - \frac{1}{2}\cos 2\theta = \boxed{-\cos 2\theta}$$

$= 0$

Section 16.5

Definition: The **del operator**, denoted by ∇, is defined as $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle$

Definition of curl and divergence:

Problem 11. Find the divergence and curl of $\mathbf{F} = \langle xy, xz, xyz^2\rangle$.

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{F} &= \frac{\partial\, xy}{\partial x} + \frac{\partial\, xz}{\partial y} + \frac{\partial\, xyz^2}{\partial z} = \\ &= y + 0 + 2xyz.\end{aligned}\right\}\ \text{divergence.}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xz & xyz^2 \end{vmatrix} =$$

$$= \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ \cdots & \cdots \end{vmatrix}\hat{\imath} - \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ \cdots & \cdots \end{vmatrix}\hat{\jmath} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ \cdots & xz \end{vmatrix}\hat{k} = \quad \text{curl}$$

$$= \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ xz & xyz^2 \end{vmatrix} \hat{i} - \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ xy & xyz^2 \end{vmatrix} \hat{j} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ xy & xz \end{vmatrix} \hat{k} =$$

$$= (xz^2 - x)\hat{i} - (yz^2 - 0)\hat{j} + (z - x)\hat{k}.$$

Theorem: If **F** is a vector field defined on all of $\Re^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then **F** is a conservative vector field. This gives us a way to determine whether a vector function on $\Re^3$ is conservative.

Problem 12. If $\mathbf{F} = \langle x,\ e^y \sin z,\ e^y \cos z \rangle$, Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{r}(t) = \langle t^4, t, 2t^2 \rangle$, for $1 \le t \le 2$.

Solution

Is $\vec{F}$ conservative?

Do the curl!

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x & e^y \sin z & e^y \cos z \end{vmatrix} =$$

$$= (e^y \cos z - e^y \cos z)\hat{i} - (0 - 0)\hat{j} + (0 - 0)\hat{k} = \vec{0}$$

so conservative!

$$\int \vec{\nabla} f \cdot d\vec{r}$$

$f_x = x \Rightarrow f = \frac{x^2}{2} + g(y,z)$

$f_y = e^y \sin z \Rightarrow f = e^y \sin z + h(x,z)$

$f_z = e^y \cos z \Rightarrow f = e^y \sin z + \ell(x,y)$

$f(x,y,z) = \frac{x^2}{2} + e^y \sin z + C$

Answer to $\int \vec{F} \cdot d\vec{r} =$

$= f(16,2,8) - f(1,1,2) =$

$= \frac{16^2}{2} + e^2 \sin 8 - \left(\frac{1}{2} + e \sin 2\right).$