Wir 11: Chapter 16: 16.1-16.9

Problem 1. Evaluate $\int_C y\,ds$, where C is parameterized by $\mathbf{r}(t) = \langle t, t^3 \rangle$, $0 \le t \le 1$.

$\langle x, y \rangle$

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

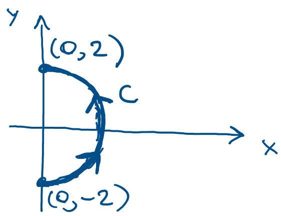
$$\int_0^1 t^3 \sqrt{1^2 + (3t^2)^2}\,dt = \int_0^1 t^3 \sqrt{1 + 9t^4}\,dt =$$

$u = 1 + 9t^4$
$du = 36t^3\,dt$

$$= \int_1^{10} \frac{1}{36} u^{1/2}\,du = \frac{1}{\cancel{36}_{18}} \frac{\cancel{2}^{1}}{3} u^{3/2}\Big|_1^{10} = \boxed{\frac{1}{54}\left(10^{3/2} - 1\right)}$$

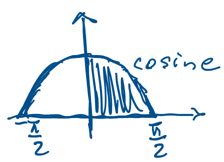
Problem 2. Find $\int_C x\,ds$, where C is the right half of the circle $x^2 + y^2 = 4$, oriented counter-clockwise.

$$\vec{r}(t) = \langle 2\cos t,\ 2\sin t \rangle,\quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$



$$\int_{-\frac{\pi}{2}}^{\pi/2} 2\cos t \sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos t\,dt = 2\int_0^{\frac{\pi}{2}} 4\cos t\,dt =$$



$$= 8\sin t\Big|_0^{\pi/2} = 8(1 - 0) = \boxed{8}$$

FTLI? $\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ z & xy & 0 \end{vmatrix} = \langle 0, --1, \dots \rangle$

not zero, so no FTLI.

$\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle z, xy, 0 \rangle$

Problem 3. Evaluate $\int_C z dx + (xy)dy$, where C is the line segment from $\overset{A}{(-1,1,0)}$ to $\overset{B}{(1,2,0)}$.

$P(t) = A + t\vec{AB}$ $\quad 0 \le t \le 1$

$\vec{r}(t) = \langle -1, 1, 0 \rangle + t\langle 2, 1, 0 \rangle =$

$\vec{r}(t) = \langle \underbrace{-1+2t}_{x}, \underbrace{1+t}_{y}, \underbrace{0}_{z} \rangle$

$dx = x'(t)\,dt = 2\,dt$ | $dy = 1\,dt$ | $dz = 0$

$$\int_0^1 \underbrace{0(2\,dt)}_{\text{zero}} + (-1+2t)(1+t)(1)dt =$$

$-1 - t + 2t + 2t^2$

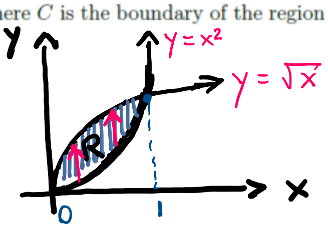
$$= \int_0^1 (-1 + t + 2t^2)\,dt =$$

$$-t + \frac{t^2}{2} + \frac{2}{3}t^3 \Big|_0^1 = -1 + \frac{1}{2} + \frac{2}{3} =$$

$$= \frac{-6+3+4}{6} = \boxed{\frac{1}{6}}$$

Green's Theorem is not a possibility here because C does not enclose a region.

Problem 4. Find $\int_C \overset{P}{(3y + 7e^{\sqrt{x}})}dx + \overset{Q}{(8x + 9\cos(y^2))}dy$, where C is the boundary of the region R enclosed by $y = x^2$ and $x = y^2$.



Green's Theorem

$$\iint_R (Q_x - P_y)\,dA$$

$$\iint_R (8-3)\,dA = 5\int_0^1 \int_{x^2}^{\sqrt{x}} 1\,dy\,dx =$$

$$= 5\int_0^1 (x^{1/2} - x^2)\,dx = 5\left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 =$$

$$= 5\int_0^1 (x^{1/2} - x^2)\,dx = 5\left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 =$$

$$= 5\left(\frac{2}{3} - \frac{1}{3}\right) = \boxed{\frac{5}{3}}$$

Problem 5. Evaluate $\int_C (xy)\,dx + (x-y)dy$, where C is the line segment from $(1,1)$ to $(2,0)$ and then from $(2,0)$ to $(3,5)$.

No Green's Theorem is possible

No FTLI either b/c $\vec{F}$ not conservative.

$\vec{r}_1(t) = \langle 1+t,\ 1-t \rangle \quad 0 \le t \le 1 \qquad C_1 \quad dx = dt,\ dy = -dt$

$\vec{r}_2(t) = \langle 2+t,\ 0+5t \rangle \quad 0 \le t \le 1 \qquad C_2 \quad dx = dt,\ dy = 5dt$

$$\int_{C_1} + \int_{C_2}$$

$$C_1 \quad \int_0^1 (1-t^2)(dt) + (\cancel{1}+\underbrace{t - \cancel{1}+t})(-dt) = \int_0^1 (1-t^2-2t)\,dt =$$

$$= t - \frac{t^3}{3} - t^2 \Big|_0^1 = \cancel{1} - \frac{1}{3} - \cancel{1} = \boxed{-\frac{1}{3} \quad C_1}$$

$$C_2 \quad \int_0^1 (2+t)(5t)(dt) + (2+t-5t)(5)\,dt =$$

$$= \int_0^1 (\underline{10t} + 5t^2 + 10 \underline{+5t} \underline{-25t})\,dt = \int_0^1 (5t^2 - 10t + 10)\,dt =$$

$$5+15 \quad \Big| \quad 20$$

$$= \frac{5t^3}{3} - 5t^2 + 10t \Big|_0^1 = \frac{5}{3} - 5 + 10 = \frac{5}{3} + 5 = \frac{5+15}{3} = \boxed{\frac{20}{3}} \quad C_2$$

Answer: $\frac{20}{3} - \frac{1}{3} = \boxed{\frac{19}{3}}$

Problem 6. A particle starts at the point $(-3, 0)$, moves along the x-axis to the point $(3, 0)$, then along the semicircle $y = \sqrt{9 - x^2}$, then back to the starting point. Find the work done on this particle by the force field $\mathbf{F} = \langle \underbrace{3x}_{P}, \underbrace{x^3 + 3xy^2}_{Q} \rangle$.

Work $= \int_C \vec{F} \cdot dr$

Green's Theorem:

$$\iint_D \Big(\underbrace{3x^2 + 3y^2}_{Q_x} - \underbrace{0}_{P_y}\Big)\, dA = \int_0^{\pi} \int_0^3 3r^2\, r\, dr\, d\theta =$$

$$3\frac{r^4}{4}\Big|_0^3 = 3\left(\frac{3^4}{4} - 0\right) = \frac{3^5}{4}$$

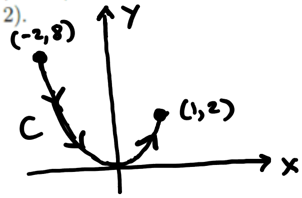
Answer: $\frac{3^5}{4}(\pi - 0) = \boxed{\frac{243}{4}\pi}$

FTLI? $\langle f_x, f_y \rangle = \langle x^2, y^2 \rangle$ POTENTIAL FUNCTION!

$\boxed{f = \frac{x^3}{3} + \frac{y^3}{3} + C}$ $\quad f(1,2) - f(-2,8)$

→ No Green's Thm is possible here, b/c the curve doesn't enclose a region.

Problem 7. Find the work done by the force field $\mathbf{F} = \langle x^2,\ y^2 \rangle$ in moving a particle along the arc of the parabola $y = 2x^2$ from the point $(-2, 8)$ to $(1, 2)$.



$\vec{r}(t) = \langle t, 2t^2 \rangle$

$-2 \le t \le 1$

$$\text{Work} = \int_C \vec{F} \cdot d\vec{r} = \int_{-2}^{1} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt =$$

C

$$= \int_{-2}^{1} (t^2 + 16t^5)\,dt =$$

-2

$\vec{F}(\vec{r}(t)) = \langle t^2, 4t^4 \rangle$, $\vec{r}\,'(t) = \langle 1, 4t \rangle$ } dot

$$= \frac{t^3}{3} + \frac{\cancel{16}^{\,8}\, t^6}{\cancel{6}_{\,3}} \Bigg|_{-2}^{1} = \frac{1}{3} + \frac{8}{3} - \left(-\frac{8}{3} + \frac{8}{3}(-2)^6\right) =$$

$2^3 \cdot 2^6$

$$= \frac{1}{3} + \frac{16}{3} - \frac{2^9}{3} = \frac{17 - 2^9}{3} = -\frac{495}{3} =$$

$$= \boxed{-165}$$

This equals $f(1,2) - f(-2,8)$

$2^4\, 2^4 \cdot 2$

$$\begin{array}{r} 16 \\ 16 \\ \hline 96 \\ 16 \\ \hline 256 \\ 2 \\ \hline 512 \\ -\ 17 \\ \hline 495 \end{array}$$

Problem 8. Given $\vec{\mathbf{F}} = \langle 4xe^z, \cos(y), 2x^2e^z \rangle$ and $\vec{\mathbf{r}}(t) = \langle \sin(t), t, \cos(t) \rangle$, compute $\int_C \mathbf{F} \cdot dr$ for $0 \le t \le \frac{\pi}{2}$.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 4xe^z & \cos y & 2x^2e^z \end{vmatrix} = \langle 0-0, -(4xe^z - 4xe^z), 0-0 \rangle$$

$\vec{F}$ is conservative

FTLI

$$\langle f_x, f_y, f_z \rangle = \langle 4xe^z, \cos y, 2x^2e^z \rangle$$

$$f_x = 4xe^z \to f = 2x^2e^z + g(y, z)$$

$$f_y = \cos y \to f = \sin y + h(x, z)$$

$$f_z = 2x^2e^z \to f = 2x^2e^z + l(y, z)$$

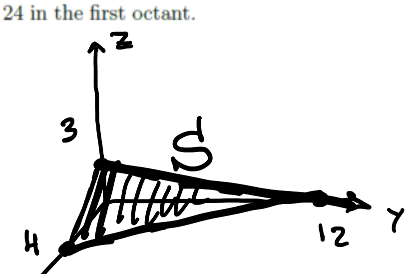
$$f(x, y, z) = 2x^2e^z + \sin y + C$$

$$\int_C \vec{F} \cdot d\vec{r} = f\left(\left(1, \tfrac{\pi}{2}, 0\right)\right) - f((0, 0, 1)) =$$

$$= (2 + 1) - (0 + 0) = \boxed{3}$$

Problem 9. Find the surface area of the part of the plane $6x + 2y + 8z = 24$ in the first octant.

$$\text{Area}(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\; dA$$

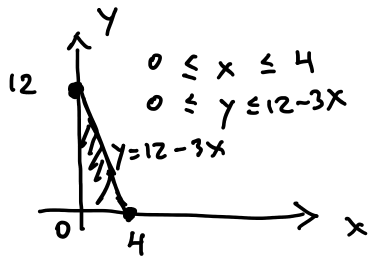


$$f(x, y) = z = \frac{24}{8} - \frac{6}{8}x - \frac{2}{8}y =$$

$$f(x,y) = z = \frac{24}{8} - \frac{6}{8}x - \frac{2}{8}y =$$

$$f = 3 - \frac{3}{4}x - \frac{1}{4}y$$

$$\int_0^4 \int_0^{12-3x} \sqrt{1 + \frac{9}{16} + \frac{1}{16}}\, dy\, dx =$$



$0 \le x \le 4$
$0 \le y \le 12-3x$

$$\int_0^4 \sqrt{\frac{26}{16}}\,(12-3x-0)\,dx =$$

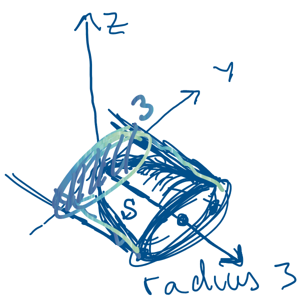
$$\frac{16+9+1}{16}$$

$$= \sqrt{\frac{13}{8}}\left[12x - \frac{3}{2}x^2\right]_0^4 =$$

$$= \frac{1}{2}\sqrt{\frac{13}{2}}\left(48 - \frac{\cancel{48}\,{}^{6}}{\cancel{2}\,{}_{1}} - 0\right) = \frac{\cancel{24}\,{}^{12}}{\cancel{2}}\sqrt{\frac{13}{2}} = \boxed{12\sqrt{\frac{13}{2}}}$$

Problem 10. Find the surface area of the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

$$f(y,z) = y^2 + z^2$$



$$\text{Area}(S) = \iint_D \sqrt{1 + (2y)^2 + (2z)^2}\, dA =$$

$$= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\; r\, dr\, d\theta =$$

$u = 1 + 4r^2$

$du = 8r\,dr$

$$\int_1^{37} \frac{1}{8} u^{1/2}\, du = \frac{1}{\cancel{8}_{4}} \cdot \frac{\cancel{2}^{1}}{3} u^{3/2}\Big|_1^{37} = \frac{1}{12}\left(37^{3/2} - 1\right)$$

$$\text{Answer: } \frac{2\pi}{12}\left(37^{3/2} - 1\right) = \boxed{\frac{\pi}{6}\left(37^{3/2} - 1\right)}$$

$$\text{Area}(S) = \iint |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

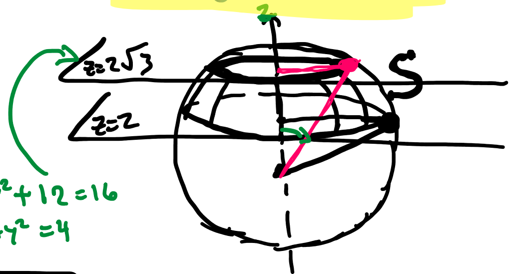
$$\boxed{\text{Area}(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, du\, dv \text{ OR } dv\, du}$$

$S : \vec{r}(u,v)$ (u,v) in D

Problem 11. Set up but do not evaluate an integral which gives is the correct set up in order to evaluate $\iint_S yz\,dS$ where S is the part of the sphere $x^2+y^2+z^2=16$ that lies between the planes $z=2$ and $z=2\sqrt{3}$. Note: If we parameterize the sphere $x^2+y^2+z^2=\rho^2$ by $\mathbf{r}(\theta,\phi)=\langle \rho\sin(\phi)\cos(\theta),\ \rho\sin(\phi)\sin(\theta),\ \rho\cos(\phi)\rangle$, then $|\mathbf{r}_\theta \times \mathbf{r}_\phi| = \rho^2\sin(\phi)$.

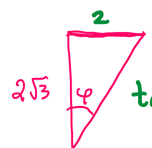
$dS = |\vec{r}_u \times \vec{r}_v|\, du\, dv$ OR $dv\, du$.

$$\left\{\begin{array}{l} 0 \le \theta \le 2\pi \\ \frac{\pi}{6} \le \varphi \le \frac{\pi}{3} \end{array}\right.$$



$x^2+y^2+12=16$
$x^2+y^2=4$

$$\boxed{\int_0^{2\pi}\int_{\pi/6}^{\pi/3} 16\sin\varphi\cos\varphi\sin\theta\,(16\sin\varphi)\,d\varphi\,d\theta}$$



$\tan\varphi = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\varphi = \frac{\pi}{6}$ ✓

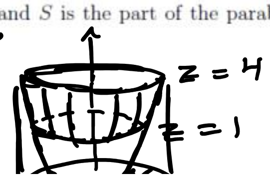
$x^2+y^2+4=16$
$x^2+y^2=12$
$r=\sqrt{4\cdot 3}=$
$=2\sqrt{3}$

$\tan\varphi = \frac{2\sqrt{3}}{2} = \sqrt{3}$

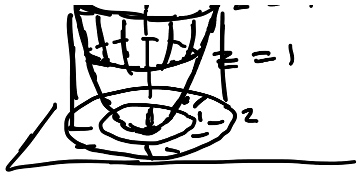
$\varphi = \frac{\pi}{3}$

Problem 12. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = \langle y,\ x,\ z\rangle$ and S is the part of the paraboloid $z = x^2+y^2$ between the planes $z=1$ and $z=4$.

4
1



$$\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$$



$$\vec{F}(\vec{r}(x,y)) = \langle y, x, x^2+y^2 \rangle$$ dot

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = \langle -2x, -2y, 1 \rangle$$

$$\underbrace{-2xy - 2yx}_{-4xy} + \underbrace{x^2+y^2}_{r^2}$$ Polar Coord.

$$\int_0^{2\pi}\int_1^2 (-4r^2 \sin\theta\cos\theta + r^2)\, r\, dr\, d\theta$$

$$\left[-r^4 \sin\theta\cos\theta + \frac{r^4}{4}\right]_{r=1}^{2} =$$

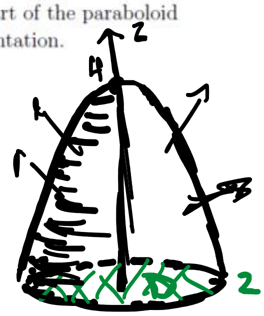
$$= -16\sin\theta\cos\theta + \frac{16}{4} - \left(-\sin\theta\cos\theta + \frac{1}{4}\right) =$$

$$-15\sin\theta\cos\theta + \frac{15}{4}$$

$$\left[-\cancel{15}\, \frac{1}{2}\sin^2\theta + \frac{15}{4}\theta\right]_0^{2\pi} = \frac{15}{4}(2\pi - 0) = \frac{15}{2}\pi.$$

Problem 13. Find the flux of $\mathbf{F} = \langle\, x,\ y,\ -z \rangle$ across S, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that is above the xy-plane. Use the positive (outward) orientation.

$$S : \vec{r}(x,y) = \langle x, y, 4 - x^2 - y^2 \rangle$$



$$\vec{F}(\vec{r}(x,y)) = \langle x, y, x^2+y^2-4 \rangle$$ dot

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} =$$

$$= \langle +2x, +2y, 1 \rangle$$ outward pointing normal vector

$\langle -f_x, -f_y, 1 \rangle$

$\langle f_x, f_y, -1 \rangle$

$$2x^2 + 2y^2 + x^2 + y^2 - 4$$

$$\int_0^{2\pi}\int_0^2 (3r^2 - 4)\, r\, dr\, d\theta$$

$$\left[\frac{3r^4}{4} - 2r^2\right]_0^2 =$$

$$= 3 \cdot \cancel{16}^{\,4} - 2 \cdot 4 - 0$$

$$= \frac{3}{\cancel{4}_1} \cdot \cancel{16}^{4} - 2 \cdot 4 - 0$$

$$(12 - 8) = 4$$

Answer: $2\pi \cdot 4 = \boxed{8\pi}$.

Given $C \longrightarrow$ plane $3x + y + z = 3$ in the first octant

Problem 14. Use Stokes' Theorem to set up but not evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle xz, 2xy, 3xy \rangle$

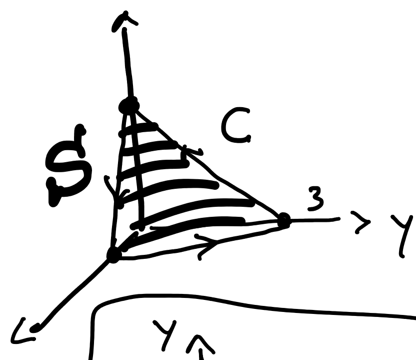
$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xz & 2xy & 3xy \end{vmatrix} =$$

$$= \langle 3x - 0, -(3y - x), 2y - 0 \rangle$$

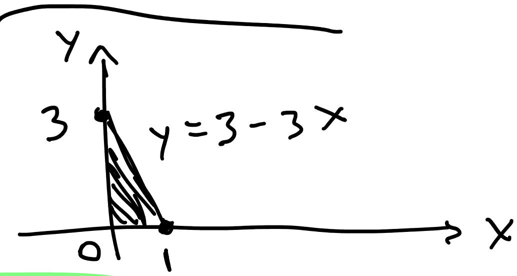
$$\langle 3x, x - 3y, 2y \rangle$$

$$S: \vec{r}(x,y) = \langle x, y, 3 - 3x - y \rangle$$



$$\operatorname{curl} \vec{F}(\vec{r}(x,y)) = \langle 3x, x - 3y, 2y \rangle$$

dot



$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{vmatrix} = \langle 3, +1, 1 \rangle$$

$$9x + x - 3y + 2y = 10x - y$$

$$\int_0^1 \int_0^{3-3x} (10x - y)\, dy\, dx \ldots$$

Problem 15. Set up but do not evaluate the integral which is the result of using Stokes' Theorem to find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle\, 2xz,\ 4x^2,\ 5y^2 \rangle$ and C is curve of intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$, oriented counterclockwise when viewed from above.

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2xz & 4x^2 & 5y^2 \end{vmatrix} = \langle 10y, +2x, 8x - 0 \rangle$$

$S: z = x + 4$

$\vec{r}(x,y) = \langle x, y, x+4 \rangle$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \langle -1, 0, 1 \rangle$$

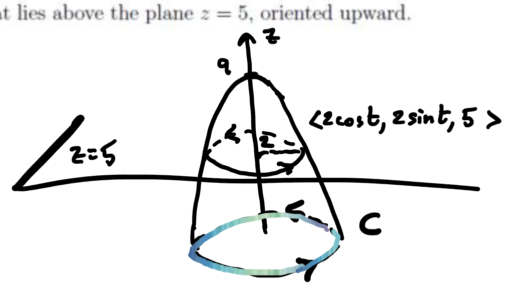
dot $\begin{cases} \langle 10y, 2x, 8x \rangle = \operatorname{curl} \vec{F}(\vec{r}(x,y)) \\ \langle -1, 0, 1 \rangle \end{cases}$

$$-10y + 8x$$

$$\int_0^{2\pi} \int_0^{2} (-10 r \sin\theta + 8 r \cos\theta)\, r \, dr \, d\theta$$

Problem 16. Use Stokes' Theorem evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle x^2 \sin(z-5), y^2, xy \rangle$ and S is the part of the paraboliod $\overset{5}{z} = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward.

$$\int_C \vec{F} \cdot d\vec{r}$$



$$\langle \underbrace{4\cos^2 t \sin(5-5)}_{0}, 4\sin^2 t, 4\cos t \sin t \rangle$$

$$\langle -2\sin t, 2\cos t, 0 \rangle$$

$$\int_0^{2\pi} 8 \sin^2 t \cos t \, dt =$$

$$= \frac{8}{3} \sin^3 t \Big|_0^{2\pi} = \boxed{0}$$

Problem 17. Using the The Divergence Theorem to evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where

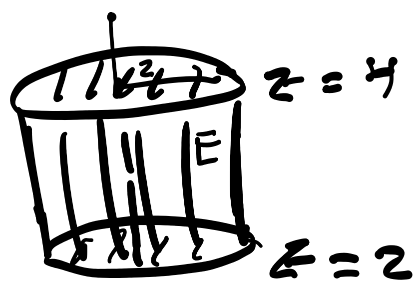
$\mathbf{F} = \left\langle 4x,\ \sin(e^z),\ \sqrt{x^3+y^2} \right\rangle$ and S is the surface bounded by $x^2+y^2=4$, $z=2$, $z=4$.

$$\iiint_E (\operatorname{div} \vec{F})\, dV =$$

$$\operatorname{div}\vec{F} = \frac{\partial 4x}{\partial x} + \frac{\partial \sin e^z}{\partial y} + \frac{\partial \sqrt{x^3+y^2}}{\partial z} =$$

$$= 4 + 0 + 0$$

$$= \iiint_E 4\, dV =$$



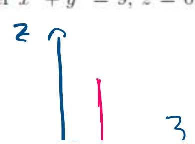
$$= \int_0^{2\pi}\int_0^2\int_2^4 4\, dz\, r\, dr\, d\theta$$

$$\left.\frac{r^2}{2}\right|_0^2 = \frac{4}{2} - 0 = 2$$

$$\boxed{32\pi}$$

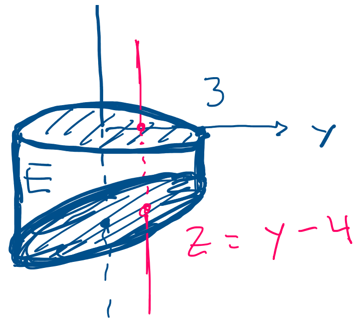
Problem 18. Using the The Divergence Theorem, find the flux of $\mathbf{F} = \left\langle ye^{z^2}, ze^x, 2z+8 \right\rangle$ across S, where S is the surface of the solid bounded by the cylinder $x^2+y^2=9$, $z=0$ and $z=y-4$.

$$\iiint (\operatorname{div}\vec{F})\, dV$$



$$\iiint_E (\operatorname{div} \vec{F})\, dV$$

$$\operatorname{div} \vec{F} = 0 + 0 + 2$$



$$\iiint_E 2\, dV =$$

$$= \int_0^{2\pi} \int_0^{3} \int_{(r\sin\theta)-4}^{0} 2\, dz\, r\, dr\, d\theta =$$

$$=$$

$$2r(4 - r\sin\theta)$$

$$2\left[2r^2 - \frac{r^3}{3}\sin\theta\right]_{r=0}^{r=3} =$$

$$2\left[18 - 9\sin\theta - 0\right]$$

$$2\left(18\theta + \underbrace{9\cos\theta}_{\text{zero}}\right)\Big|_0^{2\pi} = 2 \cdot 18(2\pi - 0)$$

$$\boxed{72\pi}$$