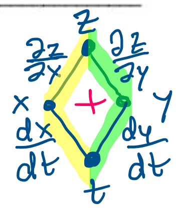
Wir 5: Sections 14.5, 14.6

Section 14.5



Problem 1. If $z = \ln(9x - 6y)$, $x = \cos(e^t)$, $y = \sin^3(4t)$, find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\left(\frac{9}{9x-6y}\right)\left(-e^t \sin e^t\right) + \left(\frac{-6}{9x-6y}\right) 3\cdot 4 \sin^2(4t)\cos(4t)$$

$$\frac{-9}{9x-6y}\left[e^t \sin e^t + 8\sin^2(4t)\cos(4t)\right]$$

$$\frac{-9}{9\cos(e^t) - 6\sin^3(4t)}\left(e^t \sin e^t + 8[\sin^2(4t)][\cos(4t)]\right).$$

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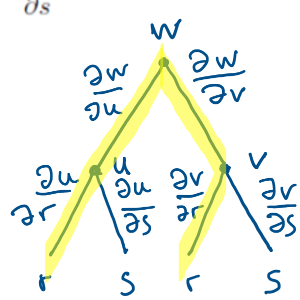
Problem 2. If $w = u^2 + 2uv$, $u = r \ln s$, $v = 2r + s$, find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial r}$$

$$(2u+2v)(\ln s) + (2u)(2)$$

$$2\left[(u+v)\ln s + 2u\right].$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$$

$$(2u+2v)\left(\frac{r}{s}\right) + (2u)1 = 2\left[(u+v)\frac{r}{s} + u\right].$$

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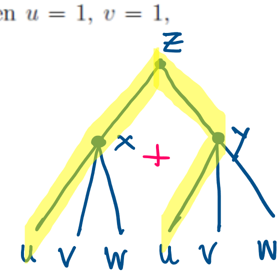
Problem 3. If $z = x^4 + xy^3$, $x = uv^3 + w^4$, $y = u + ve^w$, find find $\frac{\partial z}{\partial u}$ when $u = 1$, $v = 1$, $w = 0$.



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$(4x^3+y^3)v^3 + (3xy^2)(1) =$$

$$= (4x^3+y^3)v^3 + 3xy^2.$$

$u=1 \quad v=1 \quad w=0$

$x = 1+0 = 1$

$y = 1+1 = 2$

$$\left.\frac{\partial z}{\partial u}\right|_{\substack{u=1\\v=1\\w=0}} = (4+8)(1) + 3\cdot 1\cdot 4 =$$

$$= 12+12 = 24.$$

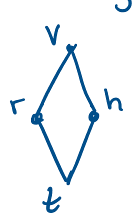
Problem 4. The height and radius of a right circular cone are changing with respect to time. If the base radius of the cone is increasing at a rate of $\frac{1}{4}$ inches per minute while the height is decreasing at a rate $\frac{1}{10}$ inches per minute, find the rate in which the volume of the cone is changing when the radius of the cone is 2 inches and the height of the cone is 1 inch.

$\frac{dr}{dt} = \frac{1}{4} \; \frac{\text{inch}}{\text{min}}$

$\frac{dh}{dt} = -\frac{1}{10} \; \frac{\text{inch}}{\text{min}}$

$\left.\frac{dV}{dt}\right|_{\substack{r=2\text{ inch}\\h=1\text{ inch}}} = ?$

$V = \frac{1}{3}\pi r^2 h$



$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} =$$

$$\left(\frac{2}{3}\pi rh\right)\frac{1}{4} + \left(\frac{1}{3}\pi r^2\right)\left(-\frac{1}{10}\right)$$

$$\frac{\cancel{2}}{3}\pi\cdot\cancel{2}\,\frac{1}{\cancel{4}} - \frac{1}{3}\pi\,\cancel{4}^{2}\cdot\frac{1}{\cancel{10}_{5}}$$

$$\frac{\pi}{3} - \frac{2}{15}\pi = \frac{3}{15}\pi = \frac{\pi}{5}\;\frac{\text{inch}^3}{\text{min}}$$

$\frac{5}{} - \frac{2}{15}$

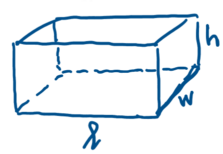
$\frac{5}{15} - \frac{2}{15}$

$\frac{\pi}{3} - \frac{2}{15}\pi = \frac{3}{15}\pi = \frac{\pi}{5} \ \frac{\text{inch}^3}{\text{min}}$

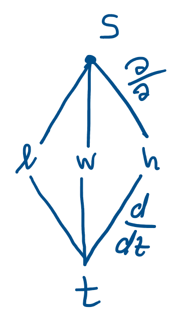
#

Problem 5. The length l, width w and height h of a box change with time. At a certain instant, the dimensions are $l = 1$ m, $w = 3$ m and $h = 2$ m, and l and w are increasing at rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that same instant, find the rate at which the surface area is changing.

$\frac{dl}{dt} = \frac{dw}{dt} = 2\ \frac{m}{s} \qquad \frac{dh}{dt} = -3\ \frac{m}{s}$



$\left.\frac{dS}{dt}\right|_{\substack{l=1\,m \\ w=3\,m \\ h=2\,m}} = ?$



$S = 2lh + 2lw + 2hw$

$\frac{dS}{dt} = \frac{\partial S}{\partial l}\frac{dl}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt}$

$(2h+2w)(2) + (2l+2w)(-3) + (2l+2h)(2)$

$(4+6)\cdot 2 - (2+6)\cdot 3 + (2+4)\cdot 2 =$

$20 - 24 + 12 = 8\ \frac{m^2}{s}$

Section 14.6

Problem 6. $f(x,y) = (xy)\sin x$, find the directional derivative at the point $\left(\frac{\pi}{2}, -1\right)$ in the direction $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

$|\vec{u}| = \sqrt{\frac{9}{25}+\frac{16}{25}} = 1$

$$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u}$$

$$\vec{\nabla} f = \langle f_x, f_y \rangle = \langle y\sin x + xy\cos x, x\sin x \rangle$$

$$\vec{\nabla} f\left(\frac{\pi}{2}, -1\right) = \langle -1+0, \frac{\pi}{2}\cdot 1 \rangle = \langle -1, \frac{\pi}{2} \rangle$$

$$D_{\vec{u}} f(P_0) = \langle -1, \frac{\pi}{2} \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = -\frac{3}{5} + \frac{\cancel{4}^{2}\pi}{\cancel{10}5} =$$

$$= \frac{2\pi - 3}{5}.$$

Problem 7. Given $f(x,y) = x^3y^2$, find the directional derivative at the point $(-1, 2)$ in the direction $4\mathbf{i} - 3\mathbf{j} = \vec{v}$

$|\vec{v}| = \sqrt{16+9} = 5$

$$\vec{u} = \langle \frac{4}{5}, -\frac{3}{5} \rangle$$

$$\vec{\nabla} f = \langle 3x^2y^2, 2x^3y \rangle \qquad \vec{\nabla} f(-1,2) = \langle 12, -4 \rangle$$

$$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u} =$$

$$= \langle 12, -4 \rangle \cdot \langle \frac{4}{5}, -\frac{3}{5} \rangle =$$

$$= \frac{48+12}{5} = \frac{60}{5} = 12$$

$$= \frac{48+12}{5} = \frac{60}{5} = 12$$

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Problem 8. If $f(x,y) = x^2 e^{xy}$, find the rate of change of f at the point $(1,0)$ in the direction of the point $P(1,0)$ to the point $Q(5,2)$. from P to Q $\vec{v} = Q - P$

$$\vec{PQ} = \vec{v} = \langle 4, 2 \rangle \qquad |\vec{v}| = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

$$\vec{u} = \frac{1}{\cancel{2}\sqrt{5}} \langle \cancel{4}^{2}, \cancel{2}^{1} \rangle = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$\vec{\nabla} f = \langle 2x e^{xy} + x^2 y e^{xy}, x^3 e^{xy} \rangle$$

$$\vec{\nabla} f(1,0) = \langle 2+0, 1 \rangle = \langle 2, 1 \rangle$$

$$D_{\vec{u}} f(1,0) = \langle 2, 1 \rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle =$$

$$= \frac{4+1}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}.$$

Problem 9. Find the gradient of $f(x,y) = x^2 + y^3 - 4xy$ at the point $(1,-1)$.

$$\vec{\nabla} f = \langle 2x - 4y, 3y^2 - 4x \rangle$$

$$\vec{\nabla} f(1,-1) = \langle 2+4, 3-4 \rangle = \langle 6, -1 \rangle.$$

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Problem 10. If $f(x,y) = x^2e^{-2y}$, $P(2,0)$, $Q(-3,1)$.

a.) Find the directional derivative at Q in the direction of P.

b.) Find a vector in the direction in which f increases most rapidly at P, and find the rate of change of f in that direction.

$\vec{QP}$

$\vec{QP} = P - Q =$

$= \langle 5, -1 \rangle$

$|\vec{QP}| = \sqrt{25+1} = \sqrt{26}$

$\vec{u} = \langle \frac{5}{\sqrt{26}}, \frac{-1}{\sqrt{26}} \rangle$

a) $D_{\vec{u}} f(Q)$

$\vec{\nabla} f = \langle 2xe^{-2y}, -2x^2e^{-2y} \rangle$

$\vec{\nabla} f(-3,1) = \langle -6e^{-2}, -18e^{-2} \rangle$

$D_{\vec{u}} f(Q) = \langle -6e^{-2}, -18e^{-2} \rangle \cdot \langle \frac{5}{\sqrt{26}}, \frac{-1}{\sqrt{26}} \rangle =$

$= \frac{-30+18}{e^2 * \sqrt{26}} = \frac{-12}{e^2\sqrt{26}}$.

$80/2$
$40/2$
$20/2$
$10/2$
$5/5$

b) direction of fastest increase is that of the gradient and the max rate of increase is the length of the gradient.

$\vec{\nabla} f(P) = \vec{\nabla} f(2,0) = \langle 4, -8 \rangle$

max increase $= \sqrt{16+64} = \sqrt{80} = 4\sqrt{5}$

Problem 11. Find the maximum rate of change of $f(x,y) = \sin^2(3x+2y)$ at the point $\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)$ and the direction in which it occurs.

The max rate of change is $\left|\vec{\nabla} f\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)\right|$

The max rate of change is $|\nabla f(\;)|$

$$\vec{\nabla} f = \langle 2\sin(3x+2y)\cos(3x+2y)\cdot 3,\ 2\sin(3x+2y)\cos(3x+2y)\cdot 2 \rangle =$$

$$= \langle 6\sin\left(\frac{\pi}{2}-\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}-\frac{\pi}{4}\right),\ 4\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) \rangle =$$

$$= \langle 6\ \frac{\sqrt{2}}{2}\ \frac{\sqrt{2}}{2},\ 4\ \frac{\sqrt{2}}{2}\ \frac{\sqrt{2}}{2} \rangle = \langle 3, 2 \rangle$$

$\Rightarrow$ max rate of change $= \sqrt{9+4} = \sqrt{13}$.

$x_0 = 1 \quad y_0 = 2 \quad z_0 = 1+4-8 = -3$

Problem 12. Find the equation of the tangent plane to the surface $f(x, y) = x^2 + y^2 - 4xy$ at the point $(1, 2)$.

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

$$z + 3 = -6(x-1) + 0(y-2)$$

$$z + 3 = -6x + 6$$

$$\boxed{6x + z - 3 = 0}$$

$f_x = 2x - 4y$

@$(1,2)$ $f_x = 2 - 8 = -6$

$f_y = 2y - 4x$

@$(1,2)$ $f_y = 4 - 4 = 0$