WIR 8

Wir 8: Sections 15.6, 15.7, 15.8

Section 15.6

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, we now define triple integrals for functions of three variables.

Definition: The **Triple Integral** of f over the box $E = \{(x, y, z) | a \le x \le b, c \le y \le d, r \le z \le s\}$ is

$$\iiint_E f(x,y,z)\,dV = \iiint_E f(x,y,z)\,dx\,dy\,dz$$

1. Evaluate $\iiint_E xyz^2\,dV$ where $E = \underset{x}{[0,1]} \times \underset{y}{[-1,2]} \times \underset{z}{[0,3]}$

$$\int_0^1 \int_{-1}^2 \left\{ \int_0^3 xyz^2\,dz \right\} dy\,dx$$

inner integral

$$xy\frac{1}{3}\left[z^3\right]_0^3 = \frac{1}{3}xy(3^3 - 0^3) = 9xy$$

middle integral

$$\int_{-1}^2 9xy\,dy = 9x\frac{1}{2}\left[y^2\right]_{-1}^2 = \frac{9}{2}x(4-1) = \frac{27}{2}x$$

outer integral

$$\int_0^1 \frac{27}{2}x\,dx = \frac{27}{2}\frac{1}{2}x^2\Big|_0^1 = \frac{27}{4}$$

#

inner integral

2. Evaluate $\int_0^1 \int_x^{x^2} \left\{ \int_x^y xyz \, dz \right\} dy dx$

inner integral

$xy \frac{1}{2}\left[z^2\right]_{z=x}^{z=y} =$

$= \frac{1}{2} xy \left(y^2 - x^2\right)$

middle integral $\int_x^{x^2} \frac{1}{2}(xy^3 - x^3y)\, dy =$

$= \frac{1}{2}\left(x \frac{1}{4}y^4 - x^3 \frac{1}{2}y^2\right)\Big|_{y=x}^{y=x^2} = \frac{1}{2}\left(\frac{1}{4}x^9 - \frac{1}{2}x^3x^4 - \frac{1}{4}x^5 + \frac{1}{2}x^5\right)$

outer integral: $\int_0^1 \frac{1}{2}\left(\frac{1}{4}x^9 - \frac{1}{2}x^7 + \frac{1}{4}x^5\right) dx = \frac{1}{2}\left(\frac{1}{4}\frac{x^{10}}{10} - \frac{1}{2}\frac{x^8}{8} + \frac{1}{4}\frac{x^6}{6}\right)\Big|_0^1 =$

$= \frac{1}{2}\left(\frac{1}{40} - \frac{1}{16} + \frac{1}{24}\right) = \frac{1}{\cancel{2}} \frac{\cancel{12 - 30 + 20}}{480} = \frac{1}{480}$

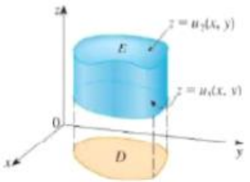
$4 \cdot 5 \quad 4 \cdot 4 \quad 6 \cdot 4$

320 -
160 -

Triple Integrals over a general bounded region E in three dimensional space:

Type I: A solid region E is said to be of type I if it lies between the graphs of two continuous functions of x and y, that is $E = \{(x,y,z) | (x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)\}$ where D is the projection of E on the xy-plane. Notice that the upper bound of E is the surface $z = u_2(x,y)$ and the lower bound of E is the surface $z = u_1(x,y)$. Moreover, it can be shown that

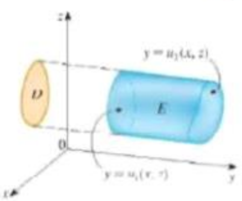
$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$



Type I $\quad dz$ — $dx\,dy$ / $dy\,dx$

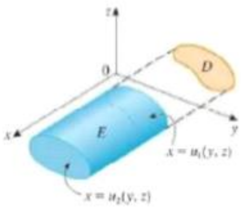
Type II: A solid region E is said to be of type II if it lies between the graphs of two continuous functions of x and z, that is $E = \{(x,y,z) | (x,z) \in D, u_1(x,z) \leq y \leq u_2(x,z)\}$ where D is the projection of E on the xz-plane. Notice that the right bound of E is the surface $y = u_2(x,z)$ and the left bound of E is the surface $y = u_1(x,z)$. Moreover, it can be shown that

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$



Type 2: $\quad dy$ — $dx\ dz$ / $dz\,dx$

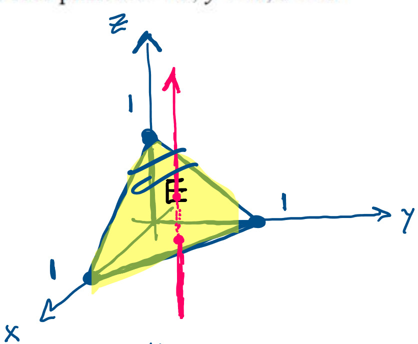
Type III: A solid region E is said to be of type III if it lies between the graphs of two continuous functions of y and z, that is $E = \{(x,y,z) | (y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\}$ where D is the projection of E on the yz-plane. Notice that the back surface of E is $x = u_1(y,z)$ and the front surface of E is the $x = u_2(y,z)$. Moreover, it can be shown that $\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dA$



Type 3 $\quad dx \begin{cases} dy\, dz \\ dz\, dy \end{cases}$

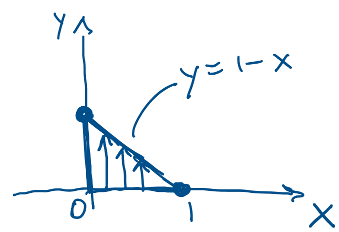
3. Evaluate $\iiint_E z\, dV$ where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. $\Rightarrow z = 1 - x - y$

$$\begin{cases} 0 \le z \le 1 - x - y \\ 0 \le x \le 1 \\ 0 \le y \le 1 - x \end{cases}$$



$$\int_0^1 \int_0^{1-x} \left\{ \int_0^{1-x-y} z\, dz \right\} dy\, dx$$

$$\frac{1}{2}\left[z^2\right]_{z=0}^{z=1-x-y} = \frac{1}{2}\left[(1-x-y)^2 - 0^2\right]$$



$$\int_0^{1-x} \frac{1}{2}(1-x-y)^2 dy = -\frac{1}{2}\,\frac{1}{3}(1-x-y)^3 \Big|_{y=0}^{y=1-x} =$$

$$= -\frac{1}{6}\left[(\cancel{1}-\cancel{x}-\cancel{1}+\cancel{x})^3 - (1-x-0)^3\right] =$$

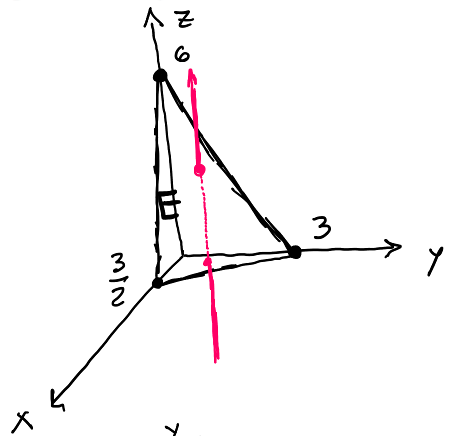
$$= +\frac{1}{6}(1-x)^3$$

$$\int_0^1 \frac{1}{6}(1-x)^3\, dx = -\frac{1}{6}\,\frac{1}{4}(1-x)^4 \Big|_0^1 = -\frac{1}{24}\left(0^4 - 1^4\right) =$$

$$= \frac{1}{24}.$$

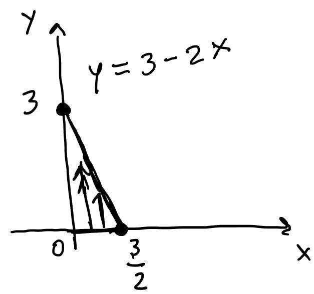
4. Evaluate $\iiint_E x\,dV$ where E is the solid bounded by the four planes $x=0$, $y=0$, $z=0$ and $4x+2y+z=6$.

$$E: \begin{cases} 0 \le z \le 6-4x-2y \\ 0 \le x \le \frac{3}{2} \\ 0 \le y \le 3-2x \end{cases}$$



$$\int_0^{3/2}\int_0^{3-2x}\left\{\int_0^{6-4x-2y} x\,dz\right\}dy\,dx$$

inner integral $x(6-4x-2y-0)$



$$\int_0^{3-2x}(6x-4x^2-2xy)\,dy =$$

$$= \left[(6x-4x^2)y - \cancel{2}x\frac{1}{\cancel{2}}y^2\right]_{y=0}^{y=3-2x} =$$

$$= (6x-4x^2)(3-2x) - x(3-2x)^2 - \emptyset =$$

$9-12x+4x^2$

$$= 18x - 12x^2 - \cancel{12x^2} + 8x^3 - 9x + \cancel{12x^2} - 4x^3$$

$$\int_0^{3/2}(4x^3-12x^2+9x)\,dx = 1x^4 - 4x^3 + \frac{9}{2}x^2\Big|_0^{3/2} =$$

$$= \frac{3^4}{2^4} - \cancel{4}\frac{3^3}{2^{\cancel{3}}} + \frac{9}{2}\frac{3^2}{2^2} - \emptyset = \frac{81}{16} - \frac{27}{2} + \frac{81}{8} =$$

162
81
243
216
27

27_8

$$= \frac{81-216+162}{16} = \frac{27}{16}.$$

5. Evaluate $\iiint_E xz\,dV$ where E is the solid tetrahedron with vertices points $(0,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,1,1)$D

A B C

$z=0$ $x=0$ $y=1$

z

$(ax+by+cz) + \cancel{d}(=0)$

$a1+b1=0$ (C)

$b1+c1=0$

$b=-a$

$b=-c$

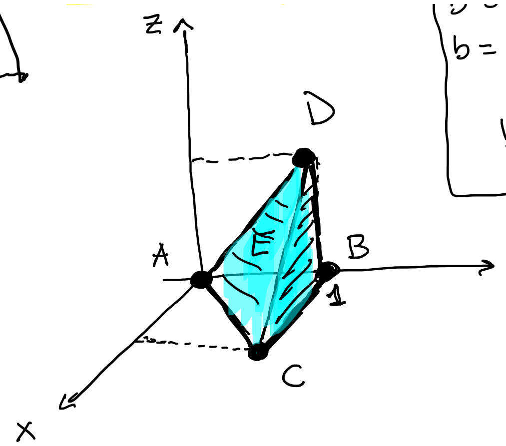
$\underbrace{a=c}$

$a=c=1$

and (0, 1, 1) D $z=0$ $x=0$ $y=1$

$\boxed{ax+by+cz} + \cancel{d} = 0$

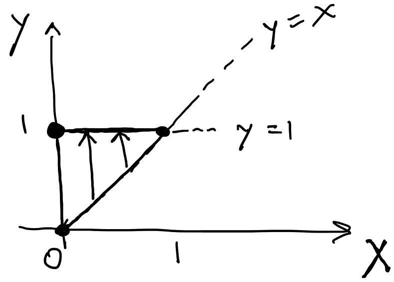
$(x+z)-y=0 \Rightarrow z=y-x$



$b=-c$ $\underbrace{a=c}$ $a=c=1$ $b=-1$

$$E:\begin{cases} 0 \le z \le y-x \\ 0 \le x \le 1 \\ x \le y \le 1 \end{cases}$$

$$\int_0^1 \int_x^1 \int_0^{y-x} xz \, dz\, dy\, dx$$



$$x \frac{1}{2}\left[z^2\right]_{z=0}^{z=y-x} =$$

$$= \frac{x}{2}\left[(y-x)^2 - \cancel{0}^2\right]$$

$$\int_x^1 \frac{x}{2}(y-x)^2 dy = \frac{x}{2}\,\frac{1}{3}(y-x)^3 \Big|_{y=x}^{y=1} = \frac{x}{6}\left[\cancel{(x-x)^3} + (1-x)^3\right]$$

$+(x-1)$

$$\frac{x}{6}\left[x^3+3x^2(-1)^1+3x(-1)^2+(-1)^3\right]$$

$$\frac{x}{6}\left(x^3-3x^2+3x-1\right)$$

$$-\int_0^1 \frac{1}{6}\left(x^4-3x^3+3x^2-x\right)dx =$$

$$= \frac{1}{6}\left(\frac{x^5}{5} - 3\frac{x^4}{4} + x^3 - \frac{x^2}{2}\right)\Big|_0^1 =$$

$$= -\frac{1}{6}\left(\frac{1}{5} - \frac{3}{4} + 1 - \frac{1}{2}\right)$$

$$= -\frac{1}{6}\left(\frac{4-15+20-10}{20}\right) = -\frac{1}{6}\left(-\frac{1}{20}\right) = \frac{1}{120}.$$

Blaise Pascal's Triangle

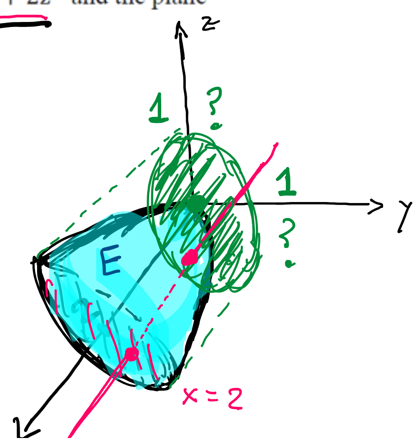
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

6. Evaluate $\iiint_E x\, dV$ where E is the 3D region bounded by the paraboloid $x = 2y^2 + 2z^2$ and the plane $x = 2$.

$2y^2+2z^2 = 2$
$r=1$

$2y^2+2z^2 \le x \le 2$

$$E:\begin{cases} 2r^2 \le x \le 2 \\ 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$$



$$\int_0^{2\pi}\int_0^1\int_{2r^2}^{2} x\, dx\, r\, dr\, d\theta$$

$$\int_0 \int_0 \int_{2r^2} x \, dx \, r \, dr \, d\theta$$

$$\frac{1}{2}\left[x^2\right]_{x=2r^2}^{x=2} = = \frac{1}{2}\left[4 - 4r^4\right]$$

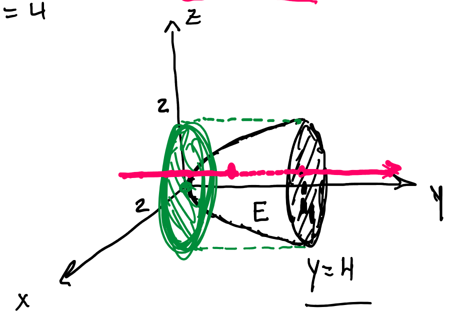
$$\int_0^1 \frac{1}{2}\left(4r - 4r^5\right) dr = \frac{1}{2}\left(2r^2 - \frac{4}{6}r^6\right)\Big|_0^1 =$$

$$= \frac{1}{2}\left(2 - \frac{2}{3} - \cancel{0}\right) = \frac{1}{\cancel{2}} \cdot \frac{\cancel{4}^{\,2}}{3} = \frac{2}{3}$$

Answer : $2\pi \, \frac{2}{3} = \frac{4}{3}\pi$

7. Evaluate $\iiint_E \overbrace{\sqrt{x^2 + z^2}}^{r} \, dV$ where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

$x^2 + z^2 = 4$



$x^2 + z^2 \le y \le 4$

$$E: \begin{cases} r^2 \le y \le 4 \\ 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$$

$$\int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \, dy \, r \, dr \, d\theta$$

$r^2(4 - r^2)$

$$\int_0^2 (4r^2 - r^4) \, dr = \frac{4}{3}r^3 - \frac{r^5}{5}\Big|_0^2 = \frac{4}{3} \cdot 8 - \frac{32}{5} =$$

$$\int_0^2 (4r^2 - r^4)\, dr = \frac{4}{3}r^3 - \frac{r^5}{5}\Big|_0^2 = \frac{4}{3}\cdot 8 - \frac{32}{5} =$$

$$= \frac{160-96}{15} = \frac{64}{15}$$

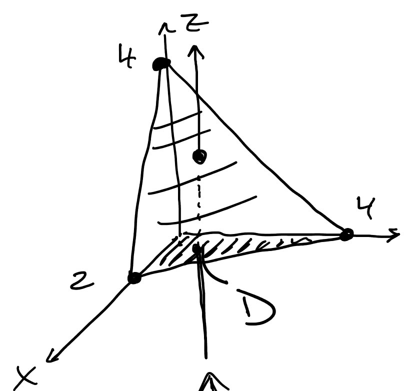
Answer: $2\pi \, \frac{64}{15} = \frac{128}{15}\pi.$

Note: We can use a triple integral to find the volume of a solid E because $Vol(E) = \iiint_E dV$.

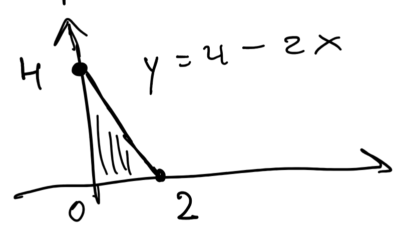
8. Consider the tetrahedron enclosed by the three coordinate planes and the plane $2x + y + z = 4$. Set up but do not evaluate:

a) a double integral that gives the volume of this solid;

b) a triple integral that gives the volume of this solid.



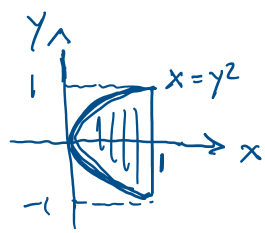
$$\left.\begin{aligned} Vol &= \iint_D f(x,y)\, dA \quad \hookrightarrow 4 - y - 2x \\ &\int_0^2 \int_0^{4-2x} (4 - 2x - y)\, dy\, dx \end{aligned}\right\} (a)$$



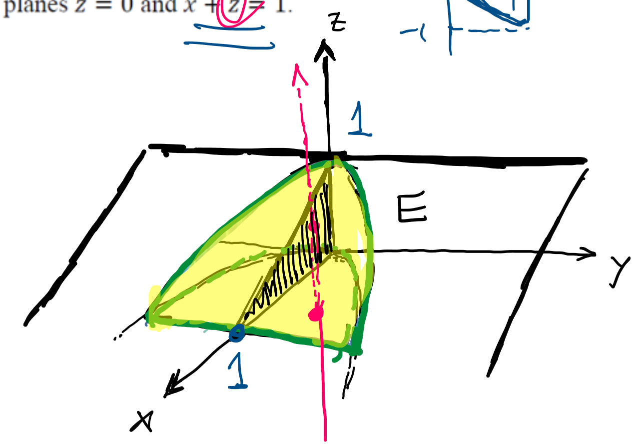
$$Vol(E) = \iiint 1\, dV = \int^{2} \int^{4-2x} \int^{4-2x-y} 1\, dz\, dy\, dx. \quad (b)$$

$$\text{Vol}(E) = \iiint_E 1\, dV = \int_0 \int_0 \int_0 1 \; dz\, dy\, dx \; .$$

9. Find the volume of the solid bounded by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$.



$$\text{Vol}(E) = \iiint_E 1\, dV =$$



$$= \int_{-1}^{1} \int_{y^2}^{1} \left\{ \int_0^{1-x} 1 \; dz \right\} dx\, dy$$

$$E: \begin{cases} 0 \le z \le 1 - x \\ -1 \le y \le 1 \\ y^2 \le x \le 1 \end{cases}$$

$$\int_{y^2}^{1} (1 - x)\, dx = x - \frac{1}{2}x^2 \Big|_{y^2}^{1} =$$

$$= \underbrace{1 - \frac{1}{2}} - y^2 + \frac{1}{2} y^4$$

$$\int^1 \quad \left(\frac{1}{2} y - \frac{y^3}{3} + \frac{1}{10} y^5 \right) \Big|^1 =$$

$$2\int_0^1\left(\frac{1}{2}-y^2+\frac{1}{2}y^4\right)dy = 2\left(\frac{1}{2}y-\frac{y^3}{3}+\frac{1}{2}\frac{y^5}{5}\right)\Big|_0^1 =$$

$$= 2\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{10}\right)-\phi =$$

$$= 2\left(\frac{15-10+3}{30}\right) = \frac{16}{30} = \frac{8}{15}$$

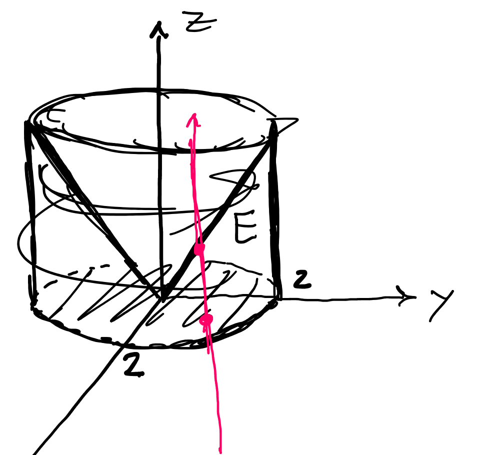
Section 15.7

10. Use cylindrical coordinates to calculate the volume above the xy-plane outside the cone $z = \sqrt{x^2+y^2}$ and inside the cylinder $x^2 + y^2 = 4$.

$z = r$

$r^2 = 4$

$$E: \begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le z \le \sqrt{x^2+y^2} \\ 0 \le z \le r \end{cases}$$



$$\text{Vol } E = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^2\int_0^r 1\,dz\,r\,dr\,d\theta$$

$\underbrace{(r-0)r}$

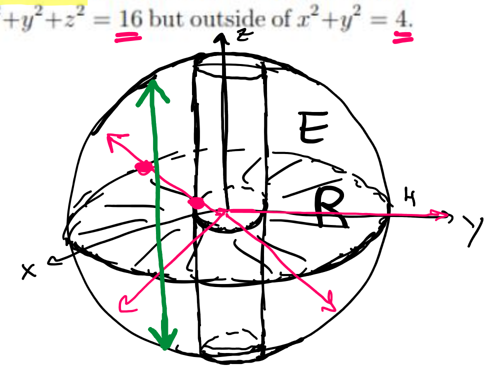
$$\int_0^2 r^2\,dr = \frac{1}{3}r^3\Big|_0^2 = \frac{1}{3}\cdot 8 - 0 = \frac{8}{3}$$

Answer $2\pi \cdot \frac{8}{3} = \frac{16}{3}\pi$

11. Consider the surfaces $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 = 4$.
Set up a triple integral in cylindrical coordinates which can be used to calculate the volume of the solid which is inside of $x^2+y^2+z^2 = 16$ but outside of $x^2+y^2 = 4$.
Calculate the volume.

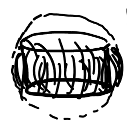
$$R: \begin{cases} 0 \le \theta \le 2\pi \\ 2 \le r \le 4 \\ -\sqrt{16-r^2} \le z \le +\sqrt{16-r^2} \end{cases}$$

$$-\sqrt{16-x^2-y^2} \le z \le +\sqrt{16-x^2-y^2}$$



Vol (napkin ring) =

$$= \int_0^{2\pi} \int_2^4 \left\{ \int_{-\sqrt{16-r^2}}^{+\sqrt{16-r^2}} 1 \, dz \, r \right\} dr \, d\theta =$$



Napkin Ring shaped region.

$$2\sqrt{16-r^2} \; r$$

$$u = 16 - r^2$$
$$du = -2r \, dr$$

$$\int_2^4 r\sqrt{16-r^2} \, dr = \int_{16-4}^{16-16} -\frac{du}{2} u^{1/2} =$$

$$= -\int_{12}^{0} \frac{1}{2} u^{1/2} \, du = \int_0^{12} \frac{1}{2} u^{1/2} \, du = \frac{1}{2}\frac{2}{3} u^{3/2} \Big|_0^{12} = \frac{1}{3}\left(12^{3/2} - \emptyset\right)$$

$$\frac{1}{3}\cdot 8\cdot 3^{3/2} = \frac{8}{3}\, 3^{3/2} = 8\sqrt{3}$$

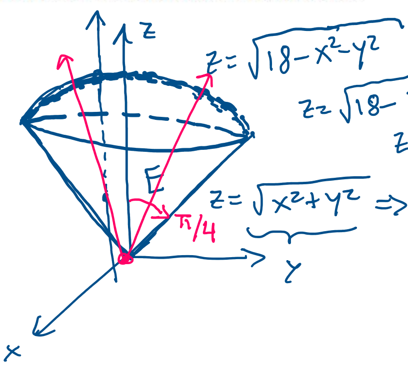
3 4

Answer: $2\pi \cdot 8\sqrt{3} = 16\pi\sqrt{3}$

12. Consider the solid shaped like an ice cream cone that is bounded by the graphs of $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{18 - x^2 - y^2}$. Set up an integral in cylindrical coordinates to find the volume of this ice cream cone.

$$\sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2}$$

$$\begin{cases} r \le z \le \sqrt{18-r^2} \\ 0 \le \theta \le 2\pi \\ 0 \le r \le \dfrac{-1+\sqrt{73}}{2} \end{cases}$$



$z = \sqrt{18-x^2-y^2}$

$z = \sqrt{18-z^2}$

$z = 18 - z^2$

$\Rightarrow z^2 + z - 18 = 0$

$\dfrac{-1 \pm \sqrt{1+72}}{2}$

$\dfrac{-1 \pm \sqrt{73}}{2}$

$z = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = z^2$

$$Vol(E) = \int_0^{2\pi}\int_0^{\frac{-1+\sqrt{73}}{2}}\int_r^{\sqrt{18-r^2}} 1 \, dz\, r\, dr\, d\theta$$

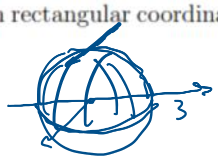
In Spherical: $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{18}} 1\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta.$

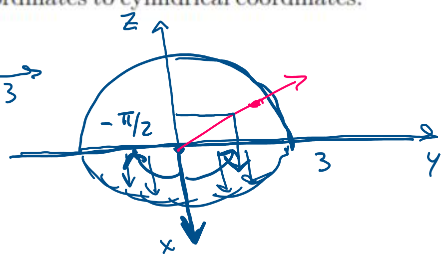
13. Consider the integral $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{\sqrt{9-x^2-y^2}}(x^2+y^2)\,dz\,dx\,dy.$

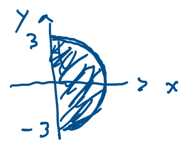
$(\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta)$

Convert the given integral from rectangular coordinates to cylindrical coordinates.

$$\begin{cases} -\frac{\pi}{2}\le\theta\le\frac{\pi}{2} \\ 0\le r\le 3 \\ 0\le z\le\sqrt{9-r^2}\end{cases}$$







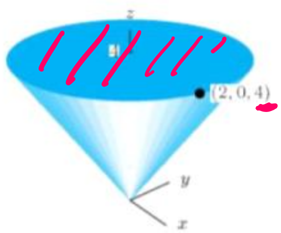
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{3}\int_{0}^{\sqrt{9-r^2}} r^2\,dz\,r\,dr\,d\theta.$$

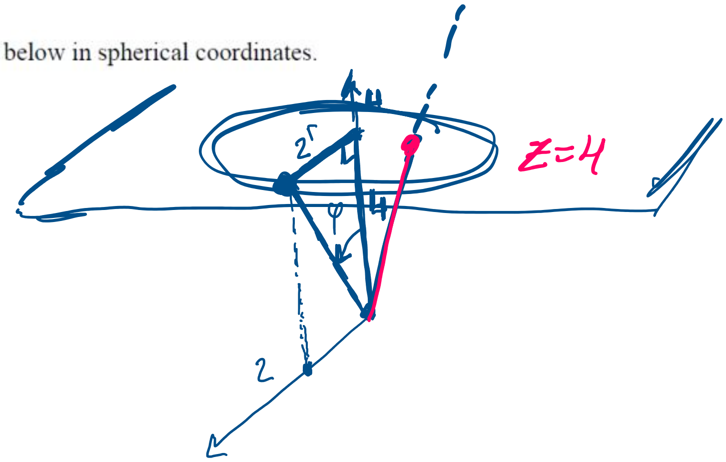
In spherical $\displaystyle\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\int_{0}^{\pi/2}\int_{0}^{3}(\rho^2\sin\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$

14. Solve problem #12 with spherical coordinates.

15. Convert the integral in problem #13 to an equivalent one in spherical coordinates.

16. Set up the volume of the region sketched below in spherical coordinates.





$z = 2\sqrt{x^2+y^2}$

$\tan\varphi = \frac{2}{4} = \frac{1}{2}$

$\varphi = \tan^{-1} 1/2$

$0 \le \rho \le$

$z = \rho\cos\varphi = 4$

$\rho = \frac{4}{\cos\varphi} = 4\sec\varphi$

$$\int_0^{2\pi}\int_0^{\tan^{-1}1/2}\int_0^{4\sec\varphi} 1\ \rho^2\sin\varphi \, d\rho \, d\varphi \, d\theta$$

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