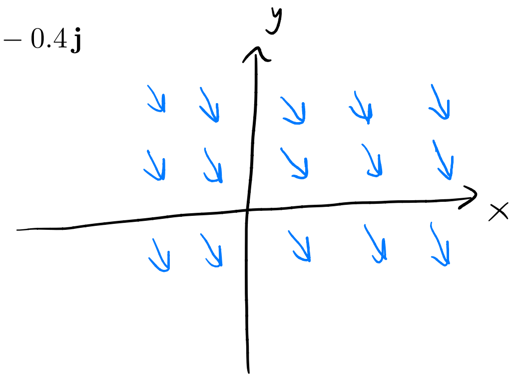
16.1 — VECTOR FIELDS

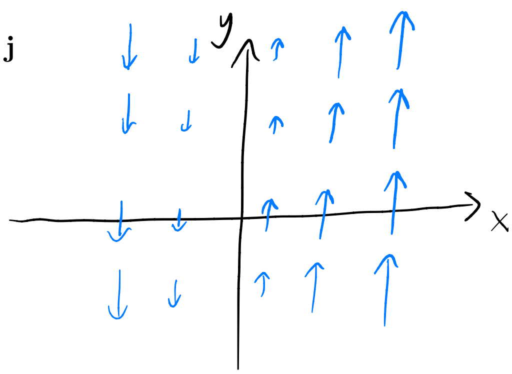
Exercise 1

Sketch the following vector fields.

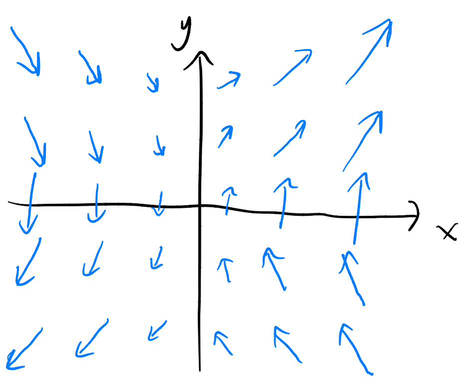
(a) $\mathbf{F} = 0.3\,\mathbf{i} - 0.4\,\mathbf{j}$



(b) $\mathbf{F} = x\,\mathbf{j}$



(c) $\mathbf{F} = \nabla f$, where $f(x, y) = xy$. $\vec{F} = \langle y, x \rangle$



16.2 — LINE INTEGRALS

Exercise 2

Compute the following line integrals.

(a) $\int_C xy^4\,\mathrm{d}s$, where C is the right half of the circle $x^2+y^2=16$.

$x = 4\cos(t)$

$y = 4\sin(t)$

$-\pi/2 \le t \le \pi/2$

$$\int_{-\pi/2}^{\pi/2} 4^5\cos(t)\sin^4(t)\sqrt{(-4\sin(t))^2+(4\cos(t))^2}\,dt$$

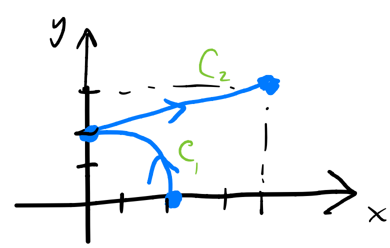
$u = \sin(t)$

$du = \cos(t)$

$$= 4^5\int_{-1}^{1} u^4\cdot 4\,dt$$

$$= 4^6\cdot\frac{1}{5}u^5\Big|_{u=-1}^{1} = \frac{4^6}{5}\left(1-(-1)\right) = \boxed{\frac{2}{5}4^6}$$

(b) $\int_C x^2\,\mathrm{d}x + y^2\,\mathrm{d}y$, where C is the arc of the circle $x^2+y^2=4$ from $(2,0)$ to $(0,2)$ followed by the line segment from $(0,2)$ to $(4,3)$.



C_1:

$x = 2\cos(t)$

$y = 2\sin(t)$ $(0 \le t \le \pi/2)$

C_2:

$\vec{r}(t) = (1-t)(0,2) + t(4,3)$

$= \langle 4t, 2+t\rangle$ $(0 \le t \le 1)$

$$\int_C x^2dx + y^2dy = \int_{C_1} x^2dx + \int_{C_1} y^2dy + \int_{C_2} x^2dx + \int_{C_2} y^2dy$$

$$= \int_0^{\pi/2} 4\cos^2(t)(-2\sin(t))\,dt + \int_0^{\pi/2} 4\sin^2(t)(2\cos(t))\,dt + \int_0^1 16t^2(4)\,dt + \int_0^1 (2+t)^2(1)\,dt$$

$$= \frac{-8}{3} + \frac{8}{3} + \frac{64}{3} + \frac{19}{3} = \boxed{\frac{83}{3}}$$

Exercise 2 continued on next page...

(c) $\int_C y^2 z \, \mathrm{d}s$, where C is the line segment from $(3, 1, 2)$ to $(1, 2, 5)$.

$$\vec{r}(t) = (1-t)(3,1,2) + t(1,2,5)$$
$$= < 3-2t, 1+t, 2+3t > \quad (0 \le t \le 1)$$

$$\int_0^1 (1+t)^2 (2+3t) \sqrt{(-2)^2 + 1^2 + 3^2} \, dt$$

$$= \sqrt{14} \int_0^1 (t^2+2t+1)(3t+2) \, dt$$

$$= \cdots$$

$$= \boxed{\frac{107}{12}\sqrt{14}}$$

(d) $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where $\mathbf{F} = \langle y, z, x \rangle$ and C is the curve given by $x = \sqrt{t}$, $y = t$, $z = t^2$, $1 \le t \le 4$.

$$\vec{r}(t) = < \sqrt{t}, t, t^2 >$$
$$\vec{r}\,'(t) = < \tfrac{1}{2} t^{-1/2}, 1, 2t >$$

$$\int_C \vec{F} \cdot \vec{r}\,'(t) \, dt$$

$$= \int_1^4 < t, t^2, \sqrt{t} > \cdot < \tfrac{1}{2} t^{-1/2}, 1, 2t > dt$$

$$= \int_1^4 \left(\tfrac{1}{2} t^{1/2} + t^2 + 2t^{3/2} \right) dt$$

$$= \cdots$$

$$= \boxed{\frac{722}{15}}$$

16.3 — FUNDAMENTAL THEOREM FOR LINE INTEGRALS

$\vec{F} = \langle P, Q \rangle$ is conservative $\Leftrightarrow \dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

Exercise 3

Check if the following vector fields are conservative or not. If they are conservative, find a potential function for the vector field.

(a) $\mathbf{F}(x, y) = \underbrace{(y^2 - 2x)}_{P}\,\mathbf{i} + \underbrace{2xy}_{Q}\,\mathbf{j}$

$$\frac{\partial Q}{\partial x} = 2y = \frac{\partial P}{\partial y} \Rightarrow \vec{F} \text{ is conservative.}$$

$$f(x,y) = \int f_x(x,y)\,dx$$
$$= \int (y^2 - 2x)\,dx$$
$$= xy^2 - x^2 + C(y)$$

$$\Rightarrow f_y(x,y) = 2xy + c'(y) = 2xy$$
$$\Rightarrow c'(y) = 0$$
$$\Rightarrow c(y) = C$$

So, $\boxed{f(x,y) = xy^2 - x^2 + C}$

value of c doesn't matter

(b) $\mathbf{F}(x, y) = \langle ye^x + \sin(y), e^x + x\cos(y) \rangle$

$$\frac{\partial Q}{\partial x} = e^x + \cos(y) = \frac{\partial P}{\partial y} \Rightarrow \vec{F} \text{ is conservative}$$

$$f(x,y) = \int f_x(x,y)\,dx$$
$$= \int (ye^x + \sin(y))\,dx$$
$$= ye^x + x\sin(y) + c(y)$$

$$f_y(x,y) = e^x + x\cos(y) + c'(y) = e^x + x\cos(y)$$
$$\Rightarrow c'(y) = 0$$
$$\Rightarrow c'(y) = C$$

So, $\boxed{f(x,y) = ye^x + x\sin(y) + C}$

value of c doesn't matter

Exercise 3 continued on next page...

(c) $\mathbf{F}(x, y) = \langle 2xy + y^{-2}, x^2 - 2xy^{-3} \rangle$ in the region where $y > 0$.

$$\frac{\partial Q}{\partial x} = 2x - 2y^{-3} = \frac{\partial P}{\partial y} \Rightarrow \vec{F} \text{ is conservative.}$$

$$f(x,y) = \int f_x(x,y)\,dx$$
$$= \int (2xy + y^{-2})\,dx$$
$$= x^2y + xy^{-2} + c(y)$$

$$f_y(x,y) = x^2 - 2xy^{-3} + c'(y) = x^2 - 2xy^{-3}$$
$$\Rightarrow c'(y) = 0$$
$$\Rightarrow c(y) = C.$$

So, $\boxed{f(x,y) = x^2y + xy^{-2} + C}$

value of c doesn't matter

(d) $\mathbf{F}(x, y) = (\ln(y) + y/x)\mathbf{i} + (\ln(x) + x/y)\mathbf{j}$

$$\frac{\partial Q}{\partial x} = \frac{1}{x} + \frac{1}{y} = \frac{\partial P}{\partial y}$$

$$f(x,y) = \int f_x(x,y)\,dx$$
$$= \int \left(\ln(y) + \frac{y}{x}\right)dx$$
$$= x\ln(y) + y\ln(x) + c(y)$$

$$f_y(x,y) = \frac{x}{y} + \ln(x) + c'(y) = \ln(x) + \frac{x}{y}$$
$$\Rightarrow c'(y) = 0$$
$$\Rightarrow c(y) = C$$

So, $\boxed{f(x,y) = x\ln(y) + y\ln(x) + C}$

value of c doesn't matter

Exercise 4

The following are all conservative vector fields. For each, find a potential function f and use it to evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

(a) $\mathbf{F}(x, y) = \langle 3 + 2xy^2, 2x^2y\rangle$. C is the arc of the hyperbola $y = 1/x$ from $(1, 1)$ to $(4, \frac{1}{4})$.

$$f(x,y) = \int (3+2xy^2)\,dx$$

$$= 3x + x^2y^2 + c(y)$$

$$\Rightarrow f_y(x,y) = 2x^2y + c'(y)$$

$$\Rightarrow c'(y) = 0$$

$$\Rightarrow c(y) = c$$

So, $\boxed{f(x,y) = 3x + x^2y^2 + c}$

$$\int_C \vec{F}\cdot d\vec{r} = f(4,\tfrac{1}{4}) - f(1,1)$$

$$= \boxed{3(4) + (4)^2(\tfrac{1}{4})^2 - 3(1) - 1}$$

(b) $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}$. C is given by $\mathbf{r}(t) = \langle \cos(t), 2\sin(t)\rangle$, $0 \le t \le \pi/2$.

$$f(x,y) = \int x^2 e^{xy}\,dy$$

$$= xe^{xy} + c(x)$$

start point: $\vec{r}(0) = \langle 1, 0\rangle$

end point: $\vec{r}(0) = \langle 0, 2\rangle$

$$\Rightarrow f_x(x,y) = e^{xy} + xye^{xy} + c'(x) = (1+xy)e^{xy}$$

$$\Rightarrow c'(x) = 0$$

$$\Rightarrow c(x) = c$$

So, $\boxed{f(x,y) = xe^{xy} + c}$

$$\int_C \vec{F}\cdot d\vec{r} = f(0,2) - f(1,0)$$

$$= 0 - 1\cdot e^0 = \boxed{-1}$$

Exercise 4 continued on next page. . .

(c) $\mathbf{F}(x,y,z) = \langle yze^{xz}, e^{xz}, xye^{xz}\rangle$. C is given by $\mathbf{r}(t) = \langle t^2+1, t+1, t^2\rangle$, $0 \le t \le 1$.

$$f(x,y,z) = \int yze^{xz}\,dx$$
$$= ye^{xz} + C(y,z)$$
$$\Rightarrow f_y(x,y,z) = e^{xz} + C_y(y,z) = e^{xz}$$
$$\Rightarrow C_y(y,z) = 0$$
$$\Rightarrow C(y,z) = C(z)$$
$$\Rightarrow f(x,y,z) = ye^{xz} + C(z)$$
$$\Rightarrow f_z(x,y,z) = xye^{xz} + C'(z) = xye^{xz}$$
$$\Rightarrow C'(z) = 0$$
$$\Rightarrow C(z) = C$$

So, $\boxed{f(x,y,z) = ye^{xz} + C}$

$$\int_C \vec{F}\cdot d\vec{r} = f(2,2,1) - f(1,1,0)$$
$$= 2e^2 - e^0$$
$$= \boxed{2e^2 - 1}$$

(d) $\mathbf{F}(x,y,z) = \langle \sin(y), x\cos(y)+\cos(z), -y\sin(z)\rangle$. C is given by $\mathbf{r}(t) = \langle \sin(t), t, 2t\rangle$, $0 \le t \le \pi/2$.

$$f(x,y,z) = \int \sin(y)\,dx$$
$$= x\sin(y) + C(y,z)$$
$$\Rightarrow f_y(x,y,z) = x\cos(y) + C_y(y,z) = x\cos(y) + \cos(z)$$
$$\Rightarrow C_y(y,z) = \cos(z)$$
$$\Rightarrow C(y,z) = y\cos(z) + C(z)$$
$$\Rightarrow f(x,y,z) = x\sin(y) + y\cos(z) + C(z)$$
$$\Rightarrow f_z(x,y,z) = -y\sin(z) + C'(z) = -y\sin(z)$$
$$\Rightarrow C'(y) = 0$$
$$\Rightarrow C(y) = C$$

So, $\boxed{f(x,y,z) = x\sin(y) + y\cos(z) + C}$

$$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(\tfrac{\pi}{2})) - f(\vec{r}(0))$$
$$= f(1, \tfrac{\pi}{2}, \pi) - f(0,0,0)$$
$$= 1\cdot\sin(\tfrac{\pi}{2}) + \tfrac{\pi}{2}\cdot\cos(\pi) - 0$$
$$= \boxed{1 - \tfrac{\pi}{2}}$$

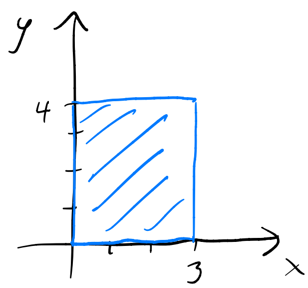
16.4 — GREEN'S THEOREM

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Exercise 5

Use Green's Theorem to evaluate the following line integrals along the given positively oriented curve.

(a) $\int_C \underbrace{ye^x}_{P}\,\mathrm{d}x + \underbrace{2e^x}_{Q}\,\mathrm{d}y$, where C is the rectangle with vertices $(0,0)$, $(3,0)$, $(3,4)$, and $(0,4)$.



$$= \iint_D (2e^x - e^x)\,dA$$

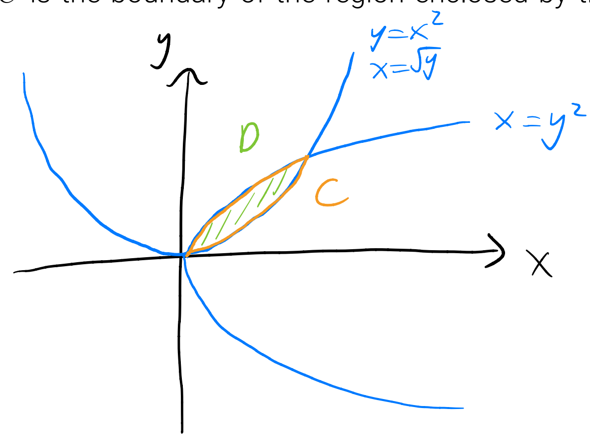
$$= \int_0^4 \int_0^3 e^x\,dx\,dy$$

$$= \int_0^4 (e^3 - 1)\,dy$$

$$= \boxed{4(e^3-1)}$$

Exercise 5 continued on next page...

(b) $\int_C (y + e^{\sqrt{x}})\,\mathrm{d}x + (2x + \cos(y^2))\,\mathrm{d}y$, where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.



$$= \iint_D (2-1)\,dA$$

$$= \iint_D dA$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} dx\,dy$$

$$= \int_0^1 (\sqrt{y} - y^2)\,dy$$

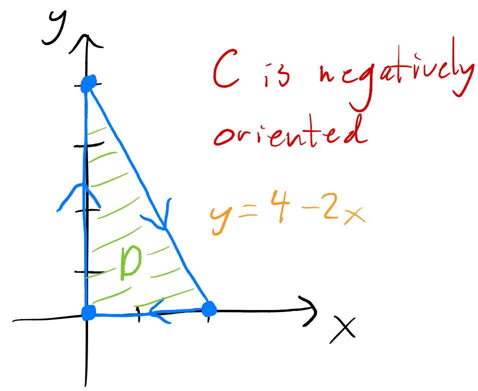
$$= \left(\frac{2}{3}y^{3/2} - \frac{1}{3}y^3\right)\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

Exercise 6

Use Green's Theorem to evaluate the following. Be sure to check the orientation of the curve.

(a) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle y\cos(x) - xy\sin(x), xy + x\cos(x)\rangle$ and C is the triangle from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ to $(0, 0)$.



$$\int_C \vec{F} \cdot d\vec{r}$$

$$= -\int_{-C} \vec{F} \cdot d\vec{r}$$

(switching the orientation so that we can apply Green's theorem)

$$= -\iint_D \left(y + \cancel{\cos(x)} - \cancel{x\sin(x)} - \left(\cancel{\cos(x)} - \cancel{x\sin(x)} \right) \right) dA$$

$$= -\iint_D y \, dA$$

$$= -\int_0^2 \int_0^{4-2x} y \, dy \, dx$$

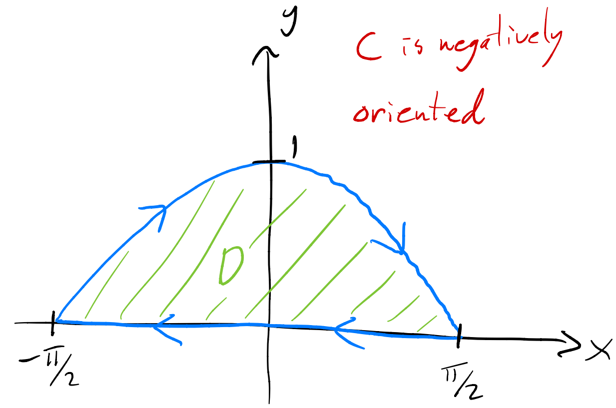
$$= -\int_0^2 \frac{1}{2} y^2 \Big|_{y=0}^{y=4-2x} dx$$

$$= -\int_0^2 \frac{1}{2}(4-2x)^2 \, dx$$

$$= \boxed{\frac{-16}{3}}$$

Exercise 6 continued on next page...

(b) $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = \langle e^{-x}+y^2, e^{-y}+x^2\rangle$ and C is the arc of the curve $y=\cos(x)$ from $(-\pi/2,0)$ to $(\pi/2,0)$ followed by the line segment from $(\pi/2,0)$ to $(-\pi/2,0)$.



C is negatively oriented

$$\int_C \vec{F}\cdot d\vec{r}$$

$$= -\int_{-C} \vec{F}\cdot d\vec{r}$$

$$= -\iint_D (2x-2y)\,dA$$

$$= -\int_{-\pi/2}^{\pi/2}\int_0^{\cos(x)} (2x-2y)\,dy\,dx$$

$$= -\int_{-\pi/2}^{\pi/2} \left.(2xy-y^2)\right|_{y=0}^{y=\cos(x)}\,dx$$

$$= -\int_{-\pi/2}^{\pi/2} \Big(\underbrace{2x\cos(x)}_{\text{use integration by parts}} - \underbrace{\cos^2(x)}_{\text{use } \cos^2(x) = \frac{1}{2}(1+\cos(2x))}\Big)\,dx$$

$$= \cdots$$

$$= \boxed{\frac{\pi}{2}}$$