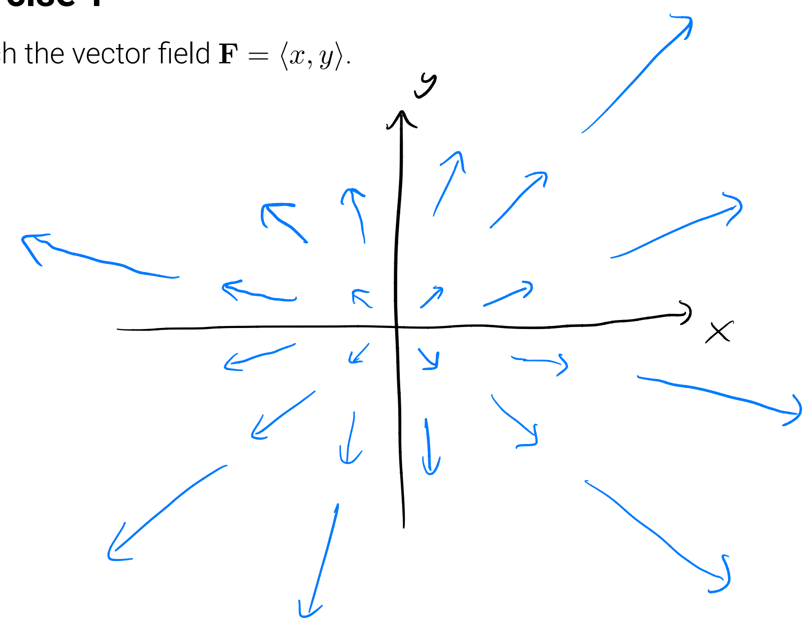
FINAL EXAM REVIEW

Exercise 1

Sketch the vector field $\mathbf{F} = \langle x, y \rangle$.

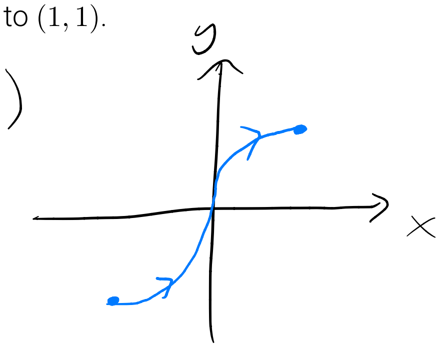


Exercise 2

Compute $\int_C e^x \, \mathrm{d}x$, where C is the arc of the curve $x = y^3$ from $(-1, -1)$ to $(1, 1)$.

$$\int_C e^x dx = \int_{-1}^{1} e^{t^3} x'(t) dt$$

$y = t$
$x = t^3$ $(-1 \le t \le 1)$



$$= \int_{-1}^{1} e^{t^3} (3t^2) dt$$

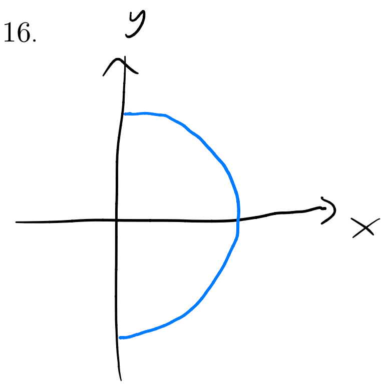
$u = t^3$
$du = 3t^2 dt$

$$= \int_{-1}^{1} e^u du$$

$$= \boxed{e - e^{-1}}$$

Exercise 3

Compute the line integral $\int_C xy^4\,\mathrm{d}s$, where C is the right half of the circle $x^2+y^2=16$.



$$\int_C xy^4\,ds = \int_C xy^4\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$x = 4\cos\theta$

$y = 4\sin\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \int_{-\pi/2}^{\pi/2} 4^5 \cos\theta\sin\theta\sqrt{16\sin^2\theta+16\cos^2\theta}\,d\theta$$

$$= 4^6\int_{-\pi/2}^{\pi/2}\cos\theta\sin^4\theta\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= 4^6\int_{-1}^{1}u^4\,du = \frac{4^6}{5}u^5\Big|_{u=-1}^{u=1} = \frac{4^6}{5}(1 - -1) = \boxed{\frac{2\cdot 4^6}{5}}$$

Exercise 4

Compute the line integral $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$, where $\mathbf{F} = \langle x+y^2, xz, y+z\rangle$ and C is given by $\mathbf{r}(t) = \langle t^2, t^3, -2t\rangle$, $0 \le t \le 2$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^2 \vec{F}(\vec{r}(t))\cdot r'(t)\,dt$$

$$= \int_0^2 \langle t^2+t^6, -2t^3, t^3-2t\rangle \cdot \langle 2t, 3t^2, -2\rangle\,dt$$

$$= \int_0^2 \left(\cancel{2t^3} + 2t^7 - 6t^5 - \cancel{2t^3} + 4t\right)dt$$

$$= \frac{2}{8}t^8 - \frac{6}{6}t^6 + \frac{4}{2}t^2\Big|_0^2$$

$$= \cancel{\frac{1}{4}(2)^8} - \cancel{(2)^6} + 2(2)^2 = \boxed{8}$$

Exercise 5

Determine whether or not $\mathbf{F}(x, y) = \langle \underbrace{xy + y^2}_{P}, \underbrace{x^2 + 2xy}_{Q} \rangle$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

$$\frac{\partial Q}{\partial x} = 2x + 2y$$

$$\frac{\partial P}{\partial y} = x + 2y$$

not equal

Therefore, $\vec{F}$ is not conservative.

Exercise 6

Determine whether or not $\mathbf{F}(x, y) = \langle \underbrace{ye^x + \sin(y)}_{P}, \underbrace{e^x + x\cos(y)}_{Q} \rangle$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

$$\frac{\partial Q}{\partial x} = e^x + \cos(y) = \frac{\partial P}{\partial y} \implies \vec{F} \text{ is conservative.}$$

$$f(x,y) = \int f_x \, dx$$

$$= \int \left(ye^x + \sin(y)\right) dx$$

$$= ye^x + x\sin(y) + C(y)$$

$$f_y = e^x + x\cos(y) + C'(y) = e^x + x\cos(y)$$

$$\Rightarrow C'(y) = 0$$

$$\Rightarrow C(y) = C$$

So, $\boxed{f(x,y) = ye^x + x\sin(y) + C}$

value of c does not matter

Exercise 7

Determine whether or not $\mathbf{F}(x,y,z) = \langle yze^{xz}, e^{xz}, xye^{xz}\rangle$ is a conservative vector field. If it is, find a potential function for the vector field $\mathbf{F}$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is a curve from $(0,0,0)$ to $(1,4,2)$.

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yze^{xz} & e^{xz} & xye^{xz} \end{vmatrix}$$

$$= \langle xe^{xz} - xe^{xz}, -(xyze^{xz} + ye^{xz} - xyze^{xz} - ye^{xz}), ze^{xz} - ze^{xz} \rangle$$

$$= \vec{0}. \Rightarrow \vec{F} \text{ is conservative.}$$

$$f(x,y,z) = \int f_x \, dx$$

$$= \int yze^{xz} \, dx$$

$$= ye^{xz} + C(y,z)$$

$$f_y = e^{xz} + C_y(y,z) = e^{xz}$$

$$\Rightarrow C_y(y,z) = 0$$

$$\Rightarrow C(y,z) = C(z)$$

$$f(x,y,z) = ye^{xz} + C(z)$$

$$f_z = xye^{xz} + C'(z) = xye^{xz}$$

$$\Rightarrow C'(z) = 0$$

$$\Rightarrow C(z) = C$$

So, $\boxed{f(x,y,z) = ye^{xz} + C}$

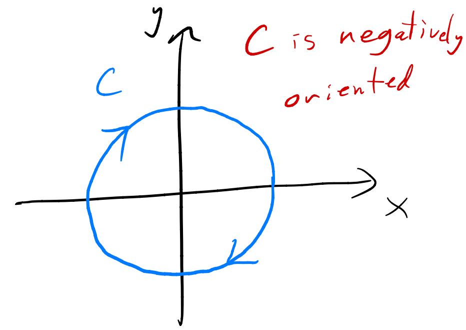
value of C does not matter

$$\int_C \vec{F}\cdot d\vec{r} = f(1,4,2) - f(0,0,0)$$

$$= \boxed{4e^2 - 0}$$

Exercise 8

Using Green's theorem, evaluate $\int_C y^3\,\mathrm{d}x - x^3\,\mathrm{d}y$, where C is the circle $x^2 + y^2 = 4$, oriented clockwise.



$$\int_C y^3 dx - x^3 dy$$

$$= -\int_{-C} y^3 dx - x^3 dy \quad (-C \text{ is positively oriented})$$

$$= -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= -\iint \left(-3x^2 - 3y^2\right) dA$$

$$= -\int_0^{2\pi}\int_0^2 -3r^2\, r\,dr\,d\theta$$

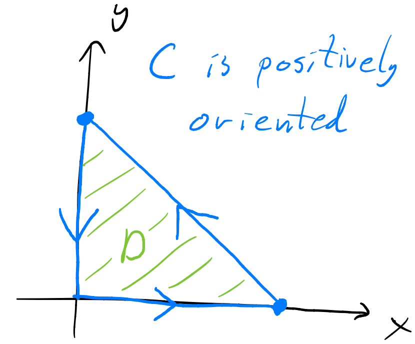
$$= 3\int_0^{2\pi} \frac{1}{4} r^4 \Big|_{r=0}^{r=2} d\theta$$

$$= 12\int_0^{2\pi} d\theta$$

$$= \boxed{24\pi}$$

Exercise 9

Using Green's theorem, find the work done by the force field $\mathbf{F} = \langle \overbrace{x(x+y)}^{P}, \overbrace{xy^2}^{Q} \rangle$ on a particle that moves from the origin along the x-axis to $(1,0)$, then along the line segment to $(0,1)$ and then back to the origin along the y-axis.



$$\text{work} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \int_0^1 \int_0^{1-y} (y^2 - x)\, dx\, dy$$

$$= \int_0^1 \left(xy^2 - \tfrac{1}{2}x^2 \right) \Big|_{x=0}^{x=1-y} dy$$

$$= \int_0^1 \left((1-y)y^2 - \tfrac{1}{2}(1-y)^2 \right) dy$$

$$= \int_0^1 \left(y^2 - y^3 - \tfrac{1}{2}(1 - 2y + y^2) \right) dy$$

$$= \int_0^1 \left(y^2 - y^3 - \tfrac{1}{2} + y - \tfrac{1}{2}y^2 \right) dy$$

$$= \int_0^1 \left(-y^3 + \tfrac{1}{2}y^2 + y - \tfrac{1}{2} \right) dy = -\frac{1}{4} + \frac{1}{6} + \frac{1}{2} - \frac{1}{2} = \boxed{\frac{-1}{12}}$$

Exercise 10

Compute the curl and divergence of the vector field $\mathbf{F} = \sin(yz)\,\mathbf{i} + \sin(zx)\,\mathbf{j} + \sin(xy)\,\mathbf{k}$.

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(yz) & \sin(zx) & \sin(xy) \end{vmatrix}$$

$$= \langle x\cos(xy) - x\cos(zx),\ y\cos(yz) - y\sin(xy),\ z\cos(zx) - z\cos(yz) \rangle$$

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}\sin(yz) + \frac{\partial}{\partial y}\sin(zx) + \frac{\partial}{\partial z}\sin(xy) = 0.$$

Exercise 11

Find a parametric representation of the following surfaces.

(a) The plane through the point $(1, 2, 1)$ that contains the vectors $\mathbf{i} - \mathbf{j}$ and $\mathbf{j} - \mathbf{k}$.

$$\vec{r}(u,v) = (1,2,1) + u(1,-1,0) + v(0,1,-1)$$

$$= \langle 1+u,\ 2-u+v,\ 1-v \rangle \qquad \begin{pmatrix} -\infty < u < \infty \\ -\infty < v < \infty \end{pmatrix}$$

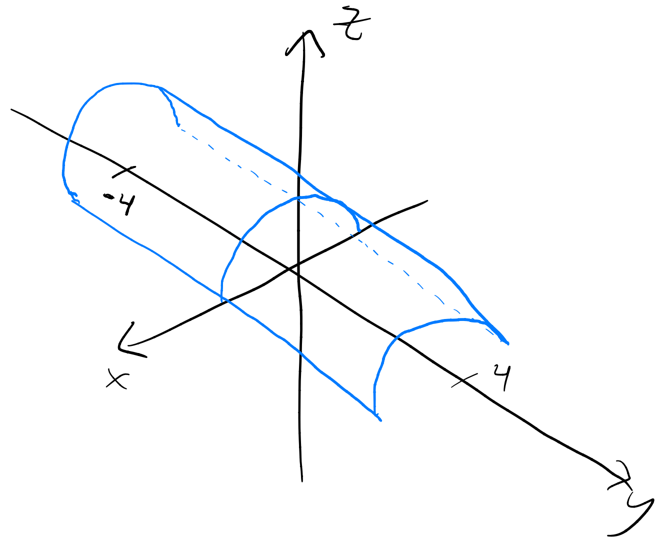
Exercise 11 continued on next page...

(b) The part of the cylinder $x^2 + z^2 = 9$ that lies above the xy-plane and between the planes $y = -4$ and $y = 4$.

$x = 3\cos\theta$

$z = 3\sin\theta$

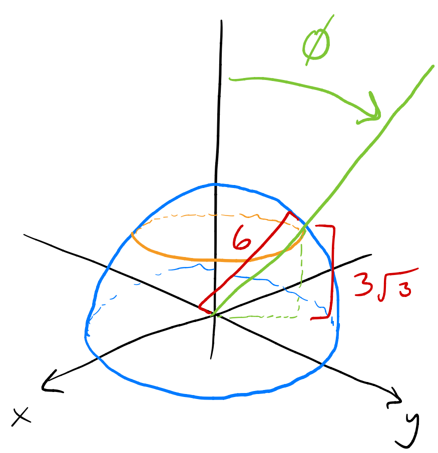
$y = y$

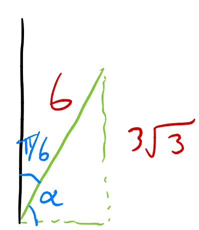


$$\boxed{\vec{r}(y,\theta) = \langle 3\cos\theta, y, 3\sin\theta \rangle \quad \begin{pmatrix} -4 \le y \le 4 \\ 0 \le \theta \le 2\pi \end{pmatrix}}$$

(c) The part of the sphere $x^2 + y^2 + z^2 = 36$ that lies between the planes $z = 0$ and $z = 3\sqrt{3}$.

$$\boxed{\begin{aligned} x &= 6\sin\phi\cos\theta \\ y &= 6\sin\phi\sin\theta \\ z &= 6\cos\phi \end{aligned} \quad \begin{pmatrix} \frac{\pi}{6} \le \phi \le \pi/2 \\ 0 \le \theta \le 2\pi \end{pmatrix}}$$





$$\alpha = \sin^{-1}\left(\frac{3\sqrt{3}}{6}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \pi/3$$

Exercise 12

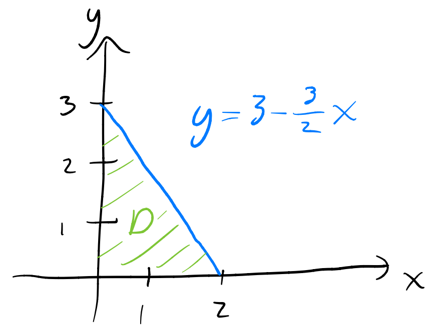
Find the surface area of the following surfaces.

(a) The part of the plane $3x + 2y + z = 6$ that lies in the first octant.

Intersection with xy-plane:

$$3x + 2y = 6$$

$$\Rightarrow y = 3 - \frac{3}{2}x$$



$$\text{surface area} = \iint_S dS$$

$$= \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

$$= \int_0^2 \int_0^{3-\frac{3}{2}x} \sqrt{3^2 + 2^2 + 1}\, dy\, dx$$

$$= \sqrt{14} \int_0^2 \int_0^{3-\frac{3}{2}x} dy\, dx$$

$$= \sqrt{14} \int_0^2 \left(3 - \frac{3}{2}x\right) dx$$

$$= \sqrt{14} \left(3x - \frac{3}{4}x^2\right)\Big|_0^2$$

$$= \sqrt{14}(6 - 3) = \boxed{3\sqrt{14}}$$

Exercise 12 continued on next page...

(b) The part of the surface $x = z^2 + y$ that lies between the planes $y = 0$, $y = 2$, $z = 0$, and $z = 2$.

Parameterize the surface:

$$\vec{r}(y,z) = \langle z^2+y, y, z\rangle \quad \begin{pmatrix} 0 \le y \le 2 \\ 0 \le z \le 2 \end{pmatrix}$$

$$\vec{r}_y = \langle 1, 1, 0\rangle$$

$$\vec{r}_z = \langle 2z, 0, 1\rangle$$

$$\vec{r}_y \times \vec{r}_z = \langle 1, -1, -2z\rangle$$

$$\text{surface area} = \iint_S dS$$

$$= \iint_D |\vec{r}_y \times \vec{r}_z|\, dA$$

$$= \int_0^2 \int_0^2 \sqrt{1^2 + (-1)^2 + (-2z)^2}\, dy\, dz$$

$$= \boxed{2\int_0^2 \sqrt{2 + 4z^2}\, dz}$$

This integral is unreasonably difficult.

Exercise 13

Compute $\iint_S y^2z^2\,\mathrm{d}S$, where S is the part of the cone $y=\sqrt{x^2+z^2}$ between the planes $y=0$ and $y=5$.

Parameterize S:

$$x = r\cos\theta$$
$$z = r\sin\theta$$
$$y = r$$

$$\begin{pmatrix} 0 \le r \le 5 \\ 0 \le \theta \le 2\pi \end{pmatrix}$$

$$\vec{r}(r,\theta) = \langle r\cos\theta, r, r\sin\theta \rangle$$

$$\vec{r}_r = \langle \cos\theta, 1, \sin\theta \rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta, 0, r\cos\theta \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \langle r\cos\theta, -r, r\sin\theta \rangle$$

$$\iint_S y^2z^2\,dS = \iint_D r^2 r^2 \sin^2\theta \,|\vec{r}_r \times \vec{r}_\theta|\,dA$$

$$= \int_0^{2\pi}\int_0^5 r^4\sin^2\theta\sqrt{r^2\cos^2\theta + r^2 + r^2\sin^2\theta}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^5 r^4\sin^2\theta\,\sqrt{2}\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^5 r^5\,dr$$

$$= \frac{\sqrt{2}}{2}\int_0^{2\pi}\left(1-\cos(2\theta)\right)d\theta\;\frac{1}{6}r^6\Big|_0^5$$

$$= \frac{\sqrt{2}\,5^6}{12}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} = \boxed{\frac{\sqrt{2}\,5^6}{6}\pi}$$

Exercise 14

Using Stokes' theorem, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle xy, yz, zx \rangle$ and C is the boundary of the part of the parabola $z = 1 - x^2 - y^2$ in the first octant. (C is oriented counterclockwise as viewed from above.)

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$, where S is the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant, oriented upward.

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 1 - r^2 \rangle \quad \begin{pmatrix} 0 \le r \le 1 \\ 0 \le \theta \le \pi/2 \end{pmatrix}$$

$$\vec{r}_r = \langle \cos\theta, \sin\theta, -2r \rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$\vec{r}_r \times \vec{r}_\theta$ gives the upward orientation, which is what we want.

$$\vec{r}_r \times \vec{r}_\theta = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle$$

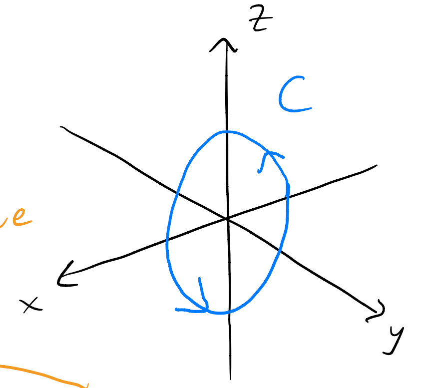
$$\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \langle -y, -z, -x \rangle$$

$$= \int_0^{\pi/2} \int_0^1 \langle -r\sin\theta, r^2 - 1, -r\cos\theta \rangle \cdot \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle \, dr \, d\theta$$

$$= \cdots = \boxed{\frac{-17}{20}}$$

Exercise 15

Using Stokes'theorem, compute $\iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}$, where $\mathbf{F} = ze^y\,\mathbf{i} + x\cos(y)\,\mathbf{j} + xz\sin(y)\,\mathbf{k}$ and S is the hemisphere $x^2+y^2+z^2=16$, $y \geq 0$, oriented in the direction of the positive y-axis.



$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$$

need the negative sign to get the orientation correct

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$\vec{r}(t) = \langle 4\cos(-t), 0, 4\sin(-t) \rangle \quad (0 \leq t \leq 2\pi)$$

$$= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$\vec{r}\,'(t) = \langle +4\sin(-t), 0, -4\cos(-t) \rangle$$

$$= \int_0^{2\pi} \langle 4\sin(-t), 4\cos(-t), 0 \rangle \cdot \langle +4\sin(-t), 0, -4\cos(-t) \rangle\,dt$$

$$= \int_0^{2\pi} 16\sin^2(-t)\,dt$$

$$= 8\int_0^{2\pi} (1 - \cos(-2t))\,dt$$

$$= 8\left(t + \tfrac{1}{2}\sin(-2t)\right)\Big|_0^{2\pi}$$

$$= \boxed{16\pi}$$

Exercise 16

Using the divergence theorem, compute the flux of $\mathbf{F} = \langle x^3 + y^3, y^3 + z^3, z^3 + x^3 \rangle$ across the surface S, where S is the sphere centered at the origin with radius 2.

$$\text{flux} = \iint_S \vec{F} \cdot d\vec{S}$$

$$= \iiint_V \operatorname{div}(\vec{F})\, dV$$

$$= \iiint_V (3x^2 + 3y^2 + 3z^2)\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} 3\rho^2 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^2 \rho^4 \, d\rho$$

$$= 3(2\pi)\left(-\cos\phi\Big|_0^{\pi}\right)\frac{1}{5}2^5$$

$$= 6\pi\left(-(-1)+1\right)\frac{1}{5}2^5$$

$$= \boxed{\frac{384\pi}{5}}$$