# 12.3 — THE DOT PRODUCT (CONTINUED...)

## Review

(a) The dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

which is also equal to

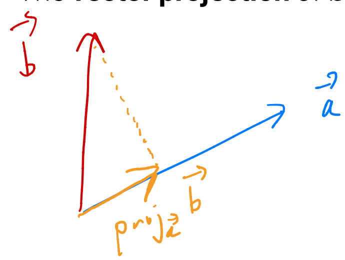
$$|\mathbf{a}||\mathbf{b}| \sin \theta.$$

(b) Two vectors are perpendicular (or orthogonal) if the angle between them is $\pi/2$ (i.e., $90°$).

(c) The zero vector $\mathbf{0}$ is orthogonal to all vectors.

(d) Two vectors are **orthogonal** if and only if their dot product is 0.

(e) The **vector projection** of $\mathbf{b}$ onto $\mathbf{a}$, denoted $\text{proj}_{\mathbf{a}}\mathbf{b}$, is given by the following picture and formula:




$$\text{proj}_{\vec{a}} \vec{b} = \left( \text{comp}_{\vec{a}} \vec{b} \right) \frac{\vec{a}}{|\vec{a}|}$$

(f) The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$, denoted $\text{comp}_{\mathbf{a}}\mathbf{b}$, is given by the formula:

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

(g) The formula for **work** is

$$\text{work} = \vec{F} \cdot \vec{d}$$

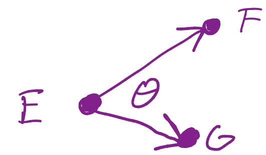
## Exercise 1

Find the angles of the triangle formed by the points $A(1, 2, 3)$, $B(-1, 2, -4)$, and $C(2, 0, -1)$.

$\vec{AB} = \langle -2, 0, -7 \rangle \quad |\vec{AB}| = \sqrt{53}$

$\vec{BC} = \langle 3, -2, 3 \rangle \quad |\vec{BC}| = \sqrt{22}$

$\vec{AC} = \langle 1, -2, -4 \rangle \quad |\vec{AC}| = \sqrt{21}$




$\vec{EF} \cdot \vec{EG} = |\vec{EF}|\,|\vec{EG}| \cos\theta$

$\theta = \cos^{-1}\left(\frac{\vec{EF} \cdot \vec{EG}}{|\vec{EF}|\,|\vec{EG}|}\right)$

$\angle CAB = \cos^{-1}\left(\frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}|\,|\vec{AC}|}\right) = \cos^{-1}\left(\frac{\langle -2, 0, -7 \rangle \cdot \langle 1, -2, -4 \rangle}{\sqrt{53}\,\sqrt{21}}\right) = \cos^{-1}\left(\frac{26}{\sqrt{53 \cdot 21}}\right).$

$\angle ABC = \cos^{-1}\left(\frac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}|\,|\vec{BC}|}\right) = \cos^{-1}\left(\frac{\langle 2, 0, 7 \rangle \cdot \langle 3, -2, 3 \rangle}{\sqrt{53}\,\sqrt{22}}\right) = \cos^{-1}\left(\frac{27}{\sqrt{53 \cdot 22}}\right).$

$\angle BCA = \cos^{-1}\left(\frac{\vec{CB} \cdot \vec{CA}}{|\vec{CB}|\,|\vec{CA}|}\right) = \cos^{-1}\left(\frac{\langle -3, 2, -3 \rangle \cdot \langle -1, 2, 4 \rangle}{\sqrt{22}\,\sqrt{21}}\right) = \cos^{-1}\left(\frac{-5}{\sqrt{22 \cdot 21}}\right).$

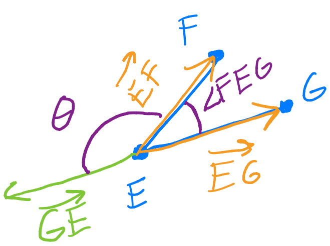
## Exercise 2

Let $E$, $F$, and $G$ be points in $\mathbb{R}^3$. Suppose $|\overrightarrow{EF}| = 3$ and $|\overrightarrow{EG}| = 4$. If $\overrightarrow{EF} \cdot \overrightarrow{EG} = 10$, then what is the angle $\angle FEG$? What is the angle between the vectors $\overrightarrow{EF}$ and $\overrightarrow{GE}$?

$10 = \vec{EF} \cdot \vec{EG} = |\vec{EF}|\,|\vec{EG}| \cos(\angle FEG)$

$= 12 \cos(\angle FEG)$

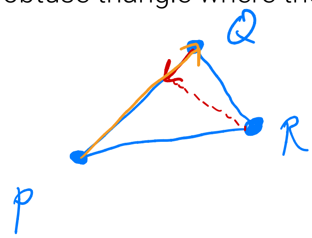
$\boxed{\angle FEG = \cos^{-1}\left(\frac{5}{6}\right)}$



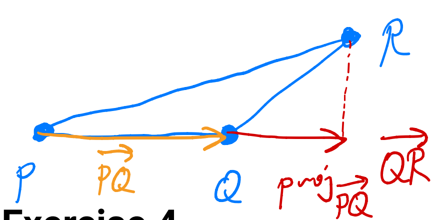

$\boxed{\theta = \pi - \angle FEG = \pi - \cos^{-1}\left(\frac{5}{6}\right)}$

## Exercise 3

Suppose the points $P, Q, R$ form an acute triangle. Is $\text{comp}_{\overrightarrow{PQ}}\overrightarrow{QR}$ positive or negative? Draw an obtuse triangle where the opposite is true.



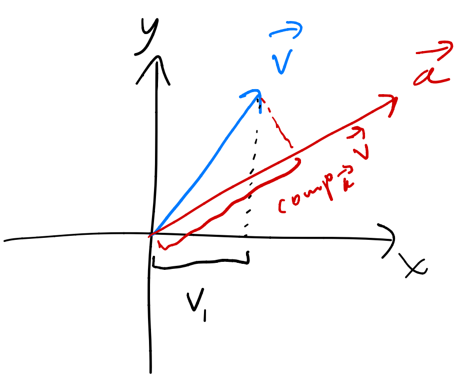

$\text{proj}_{\overrightarrow{PQ}}\overrightarrow{QR}$ and $\overrightarrow{PQ}$ are pointing in opposite directions, so $\text{comp}_{\overrightarrow{PQ}}\overrightarrow{QR}$ is negative.

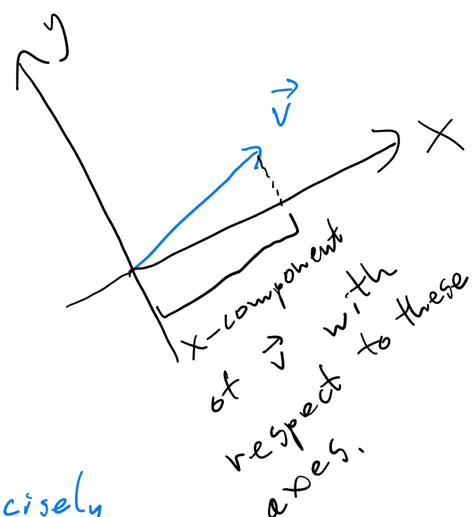



$\text{proj}_{\overrightarrow{PQ}}\overrightarrow{QR}$ and $\overrightarrow{PQ}$ are pointing in the same direction, so $\text{comp}_{\overrightarrow{PQ}}\overrightarrow{QR}$ is positive.

## Exercise 4

Let $\mathbf{v} = (v_1, v_2, v_3)$. Then, we call $v_1$ the "$x$-component of $\mathbf{v}$". Explain why it makes sense that people often call $\text{comp}_{\mathbf{a}}\mathbf{v}$ the "$\mathbf{a}$-component of $\mathbf{v}$".




The $\vec{a}$-component of $\vec{v}$ is precisely what the x-component of $\vec{v}$ would be if the x-axis pointed in the direction of $\vec{a}$.

# 12.4 — THE CROSS PRODUCT

## Review

(a) The cross product of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

which is also equal to

$$|\vec{a}|\,|\vec{b}| \sin\theta.$$

(b) The result of the cross product is a vector that points in the direction given by the right hand rule.

(c) $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Nonzero (d) Two vectors $\mathbf{a}$ and $\mathbf{b}$ are ~~perpendicular~~ parallel if and only if $\vec{a} \times \vec{b}$.

(e) $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$. However, we have

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

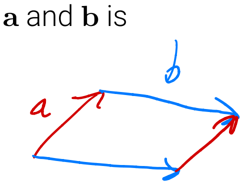
(f) The order in which we do cross products matters: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

(g) Cross products of the standard basis vectors:

$$\mathbf{i} \times \mathbf{j} = \vec{k} \qquad \mathbf{j} \times \mathbf{i} = -\vec{k}$$
$$\mathbf{j} \times \mathbf{k} = \vec{i} \qquad \mathbf{k} \times \mathbf{j} = -\vec{i}$$
$$\mathbf{k} \times \mathbf{i} = \vec{j} \qquad \mathbf{i} \times \mathbf{k} = -\vec{j}$$

(h) The **area** of a parallelogram formed by the vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$|\vec{a} \times \vec{b}|.$$



(i) The **volume** of a parallelepiped formed by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is given by

$|\vec{a}\cdot(\vec{b}\times\vec{c})|$.

(j) **Torque** is given by the formula

$$\vec{\tau} = \vec{r} \times \vec{F}$$

(k) The direction of the torque vector tells you the axis of rotation.

## Exercise 5

Compute $\langle 1, 5, -2\rangle \times \langle -2, 1, 3\rangle$.

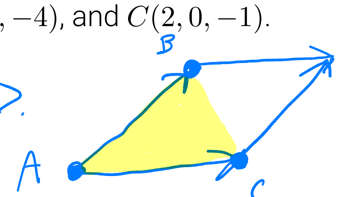
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & -2 \\ -2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 5 \\ -2 & 1 \end{vmatrix}\vec{k}$$

$$= 17\vec{i} + \vec{j} + 11\vec{k}.$$

## Exercise 6

Find the area of the triangle formed by the points $A(1, 2, 3)$, $B(-1, 2, -4)$, and $C(2, 0, -1)$.

$\vec{AB} = \langle -2, 0, -7\rangle$, $\vec{AC} = \langle 1, -2, -4\rangle$.




area of $\Delta ABC = \frac{1}{2}$(area of parallelogram)

$$= \frac{1}{2}\left|\det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & -7 \\ 1 & -2 & -4 \end{pmatrix}\right|$$

$$= \frac{1}{2}\left|-14\vec{i} - 15\vec{j} + 4\vec{k}\right|$$



$$\boxed{= \frac{1}{2}\sqrt{14^2 + 15^2 + 4^2}.}$$

## Exercise 7

Find two unit vectors that are orthogonal to the plane that passes through the points $P(1,0,1)$, $Q(2,3,4)$, and $R(2,1,1)$.

$\overrightarrow{PQ} = \langle 1, 3, 3 \rangle \qquad \overrightarrow{PR} = \langle 1, 1, 0 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 3 \\ 1 & 1 & 0 \end{vmatrix} = \langle -3, +3, -2 \rangle.$$

$$|\langle -3, 3, -2 \rangle| = \sqrt{3^2+3^2+2^2} = \sqrt{22}.$$

$$\boxed{\left\langle \frac{-3}{\sqrt{22}}, \frac{3}{\sqrt{22}}, \frac{-2}{\sqrt{22}} \right\rangle \text{ and } \left\langle \frac{3}{\sqrt{22}}, \frac{-3}{\sqrt{22}}, \frac{2}{\sqrt{22}} \right\rangle.}$$

# 12.5 – LINES AND PLANES

## Review

(a) A **vector** (or **parametric**) equation for a line is of the form

$$\vec{r}(t) = \vec{r}_0 + t\,\vec{v}.$$

(b) Taking the components of this equation, we obtain the parametric equations of the line:

$$x = x_0 + t v_1$$
$$y = y_0 + t v_2$$
$$z = z_0 + t v_3$$

(c) The **symmetric equations** of a line are of the form

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}.$$

(d) Skew lines are lines that do not intersect and are not parallel.

(e) The standard form for a line segment is

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\,\vec{r}_1.$$

## Exercise 8

Find the parametric and symmetric equations for the line that goes through the points $(6, 2, -1)$ and $(2, -4, 1)$.

$\vec{v} = \langle 6-2, 2-{-4}, -1-1 \rangle = \langle 4, 6, -2 \rangle$.

$\vec{r}(t) = \langle 6, 2, -1 \rangle + t\,\vec{v}$

$= \langle 6+4t, 2+6t, -1-2t \rangle$

$$\boxed{\begin{aligned} x &= 6+4t \\ y &= 2+6t \\ z &= -1-2t \end{aligned}} \quad \begin{aligned} &\Rightarrow t = \frac{x-6}{4} \\ &\Rightarrow t = \frac{y-2}{6} \\ &\Rightarrow t = \frac{z+1}{-2} \end{aligned} \quad \boxed{\frac{x-6}{4} = \frac{y-2}{6} = \frac{z+1}{-2}}$$

## Exercise 9

For this problem, use the line you found in the previous exercise.

(a) Does the line go through the point $(-2, -8, 3)$?

Plug into the symmetric equations:

$\frac{-2-6}{4} \neq \frac{-8-2}{6} \neq \frac{3+1}{2}$. Since the point does not satisfy the symmetric equations, it is not on the line.

(b) Where does the line intersect the plane $y = 100$?

$100 = 2+6t \Rightarrow t = \frac{98}{6} = \frac{49}{3}$.

$\vec{r}\left(\frac{49}{3}\right) = \boxed{\left\langle 6+4\left(\frac{49}{3}\right), 100, -1-2\left(\frac{49}{3}\right) \right\rangle}$.

## Exercise 10

Find parametric equations for the line $\frac{x-3}{2} = \frac{y+6}{7} = \frac{z-2}{3}$.

$$\frac{x-3}{2} = t \Rightarrow x = 3 + 2t$$

$$\frac{y+6}{7} = t \Rightarrow y = -6 + 7t$$

$$\frac{z-2}{3} = t \Rightarrow z = 2 + 3t$$

## Exercise 11

Determine if the following pair of lines is intersecting, parallel, or skew: $L_1(t) = \langle 1+t, -2-3t, 6+2t \rangle$ and $L_2(t) = \langle 4+t, -6+2t, 8-2t \rangle$.

If they intersect, then there must be $t$ and $s$ such that

$$< 1+t, -2-3t, 6+2t > = < 4+s, -6+2s, 8-2s >.$$

$$1+t = 4+s \Rightarrow t = 3+s$$

$$-2-3t = -6+2s \Rightarrow -2-3(3+s) = -6+2s$$

$$\Rightarrow -2-9-3s = -6+2s$$

$$\Rightarrow s = -1 \Rightarrow t = 2$$

$$L_1(2) = < 3, -8, 10 > = L_2(-1) = < 3, -8, 10 >$$

Therefore, $L_1$ and $L_2$ intersect at $(3, -8, 10)$.

## Exercise 12

Find the parametric and symmetric equations for a line passing through $(5, -2, 1)$ that is parallel to the line $L(t) = \langle 6 - 2t, 5 + 4t, -2 - t \rangle$.

L goes in the direction $\langle -2, 4, -1 \rangle$.

Plugging into the equation for a line:

$$\boxed{\langle 5, -2, 1 \rangle + t \langle -2, 4, -1 \rangle}$$

## Exercise 13

Find a parametric equation for the line segment that starts at $A(3, -1, -2)$ and goes 5 units in the direction $\langle 2, 2, 1 \rangle$.

$$|\langle 2, 2, 1 \rangle| = \sqrt{2^2 + 2^2 + 1^2} = 3.$$

So, the unit vector in the $\langle 2, 2, 1 \rangle$ direction is $\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$.

So, the other endpoint of the line segment is

$$\left(3 + 5\left(\tfrac{2}{3}\right), -1 + 5\left(\tfrac{2}{3}\right), -2 + 5\left(\tfrac{1}{3}\right)\right) = \left(\tfrac{26}{3}, \tfrac{14}{3}, \tfrac{10}{3}\right).$$

Plugging into the equation for a line segment:

$$\boxed{(1-t)\langle 3, -1, -2 \rangle + t \langle \tfrac{26}{3}, \tfrac{14}{3}, \tfrac{10}{3} \rangle \quad (0 \le t \le 1)}$$