14.6 — DIRECTIONAL DERIVATIVES AND THE GRADIENT

Review

(a) The **gradient** of a function $f(x, y)$ is

$$\nabla f(x,y) = \left(f_x(x,y), f_y(x,y) \right).$$

(b) The **gradient** of a function $f(x, y, z)$ is

$$\nabla f(x,y,z) = \left(f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \right).$$

(c) The **directional derivative** of $f(x, y)$ at the point (a, b) in the direction $\vec{u}$ is the slope of the function f in the direction $\vec{u}$. It is given by the formula

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}.$$

(d) The gradient points in the direction of maximal slope.

(e) The magnitude of the gradient is the maximal slope.

(f) The gradient is perpendicular to the level curves.

Exercise 1

Compute the gradient of the following functions.

(a) $f(x, y) = x \cos(xy)$

$$\nabla f(x,y) = \left(-yx\sin(xy) + \cos(xy), -x^2\sin(xy) \right).$$

(b) $g(x, y, z) = \sin(xy)e^{yz} + zy^2$

$$\nabla g(x,y,z) = \left(y\cos(xy)e^{yz}, z\sin(xy)e^{yz} + x\cos(xy)e^{yz} + 2yz, y\sin(xy)e^{yz} + y^2 \right)$$

(c) $h(x, y, z, w) = e^{xyz} + w^x$

$$\nabla h(x,y,z,w) = \left(yze^{xyz} + \ln(w)w^x, xze^{xyz}, xye^{xyz}, xw^{x-1} \right)$$

Exercise 2

Find the following directional derivatives of $f(x, y) = \sec(xy) + x^2$ at $(2, 1)$ in the direction towards the point $(5, 5)$.

$$f_x(x,y) = y\sec(xy)\tan(xy) + 2x$$

$$f_y(x,y) = x\sec(xy)\tan(xy)$$

$$f_x(2,1) = \sec(2)\tan(2) + 2$$

$$f_y(2,1) = 2\sec(2)\tan(2)$$

$$\vec{v} = (5-2, 5-1) = (3,4)$$

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$\boxed{D_{\vec{u}} f(2,1) = \frac{3}{5}\left(\sec(2)\tan(2) + 2\right) + \frac{4}{5}\cdot 2\sec(2)\tan(2)}$$

Exercise 3

Find the line tangent to $g(x, y) = e^{x^2+y}$ at $(x, y) = (-1, 1)$ that goes in the same direction as $\langle 5, 12\rangle$.

$$\nabla g(x,y) = \left(2xe^{x^2+y}, e^{x^2+y}\right).$$

$$\vec{u} = \left(\frac{5}{13}, \frac{12}{13}\right)$$

$$g(-1,1) = e^{(-1)^2+1} = e^2.$$

$$D_{\vec{u}} g(-1,1) = \nabla g(-1,1)\cdot\vec{u} = \left(2(-1)e^2, e^2\right)\cdot\left(\frac{5}{13}, \frac{12}{13}\right) = \frac{-10}{13}e^2 + \frac{12}{13}e^2$$

$$= \frac{2}{13}e^2.$$

$$\boxed{L(t) = (-1, 1, e^2) + t\left(\frac{5}{13}, \frac{12}{13}, \frac{2}{13}e^2\right).}$$

Exercise 4

Sketch the level curves of the function and draw the gradient at several points.

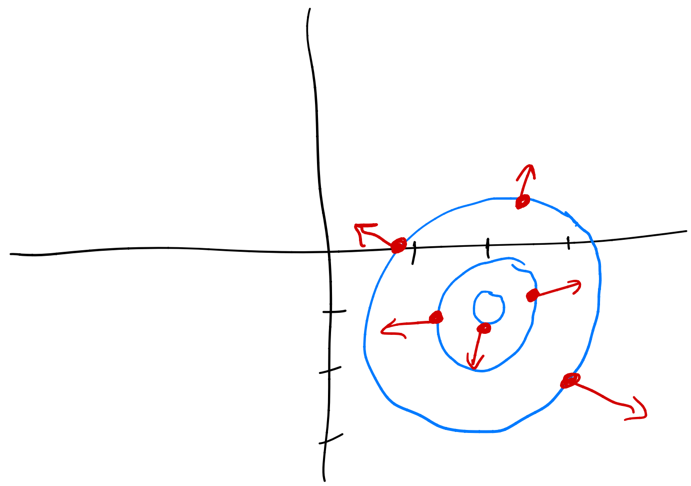
(a) $f(x, y) = e^{(x-2)^2+y^2+2y}$

$e^{(x-2)^2+y^2+2y} = c$

$(x-2)^2+y^2+2y+1 = \ln(c)+1$

$(x-2)^2+(y+1)^2 = \ln(c)+1$

circles with center $(2,-1)$

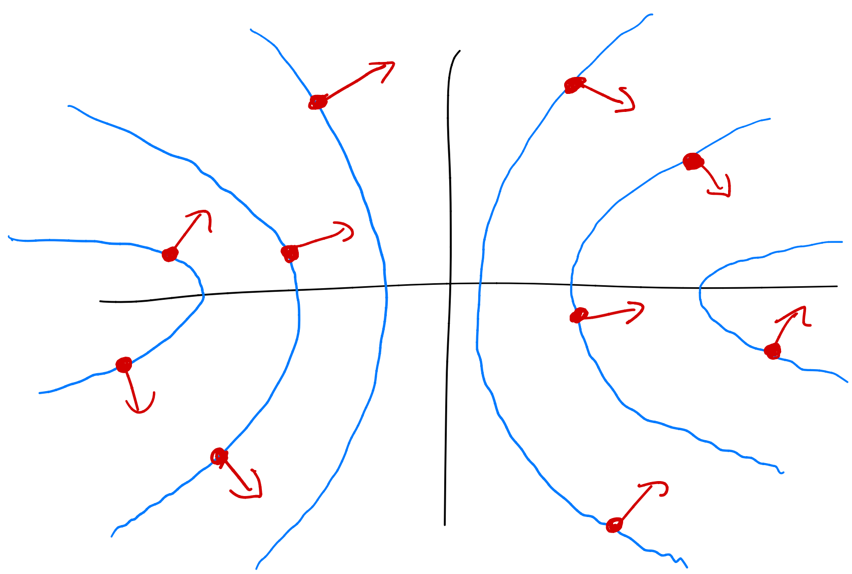


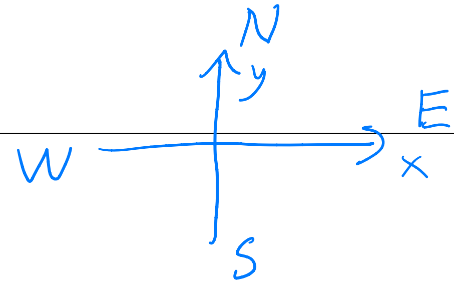
(b) $g(x, y) = \dfrac{x}{y^2+2}$

$\dfrac{x}{y^2+2} = c$

$x = cy^2+2c$

parabolas opening sideways





Exercise 5

Suppose you are climbing a hill whose shape is given by the equation $z = 1000 - 0.005x^2 - 0.01y^2$, and you are standing at the point $(60, 40, 966)$. The positive x-axis points east and the positive y-axis points north.

(a) If you walk due south, will you start to ascend or descend? At what rate?

$\nabla f(x,y) = (-0.01x, -0.02y)$ $\quad \nabla f(60,40) = (-0.6, -0.8)$

$\vec{S} = (0,-1)$ $\quad D_{\vec{S}} f(60,40) = \nabla f(60,40) \cdot \vec{S} = 0.8.$

Since the directional derivative is > 0, you will ascend with slope 0.8.

(b) If you walk northwest, will you start to ascend or descend? At what rate?

$\vec{NW} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$

$D_{\vec{NW}} f(60,40) = (-0.6, -0.8) \cdot \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{0.6}{\sqrt{2}} - \frac{0.8}{\sqrt{2}} = \frac{-0.2}{\sqrt{2}}$

Since the directional derivative is < 0, you will descend with slope $\frac{-0.2}{\sqrt{2}}$.

(c) In which direction is the slope largest? What is the rate of ascent in that direction? At what angle above the horizontal does the path in that direction begin?

The slope is largest in the direction of the gradient, i.e., $(-0.6, -0.8)$. The slope in that direction is $|(-0.6, 0.8)| = 1$. The angle is $\tan^{-1}(1) = 45\%$.

Exercise 6

Use the gradient vector to find the tangent line to the level curve of $f(x,y) = x^2 + y^2 - 4x$ at the point $(1,2)$.

$f(1,2) = 1$

$\nabla f(x,y) = (2x-4, 2y)$.

$\nabla f(1,2) = (-2, 4)$.

The level curve is $\perp$ to the gradient, so it goes in the direction $(4,2)$.

$$\boxed{L(t) = (1,2,1) + t(4,2,0).}$$

Exercise 7

Let $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ and $\mathbf{v} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$. Given that $D_{\mathbf{u}} f(0,0) = 3$ and $D_{\mathbf{v}} f(0,0) = -2$, find $\nabla f(0,0)$.

$$3 = D_{\vec{u}} f(0,0) = \nabla f(0,0) \cdot \vec{u} = \frac{3}{5} f_x(0,0) + \frac{4}{5} f_y(0,0)$$

$$-2 = D_{\vec{v}} f(0,0) = \nabla f(0,0) \cdot \vec{v} = \frac{1}{\sqrt{2}} f_x(0,0) + \frac{1}{\sqrt{2}} f_y(0,0)$$

$$-2\left(\frac{3\sqrt{2}}{5}\right) = \frac{3}{5} f_x(0,0) + \frac{3}{5} f_y(0,0)$$

$$\Rightarrow 3 + \frac{6\sqrt{2}}{5} = \frac{1}{5} f_y(0,0) \Rightarrow \boxed{f_y(0,0) = 15 + 6\sqrt{2}}$$

$$-2\left(\frac{4\sqrt{2}}{5}\right) = \frac{4}{5} f_x(0,0) + \frac{4}{5} f_y(0,0)$$

$$\Rightarrow -\frac{8\sqrt{2}}{5} - 3 = \frac{1}{5} f_x(0,0) \Rightarrow \boxed{f_x(0,0) = -15 - 8\sqrt{2}}$$

14.7 — MAXIMUM AND MINIMUM VALUES

Review

(a) **Critical points** of a function f are the points where

$\nabla f = 0$.

(b) **Second derivatives test:** Suppose f is a function of two variables with continuous second order partial derivatives and $f_x(a,b) = 0 = f_y(a,b)$. Define

$$D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \Big(f_{xy}(a,b)\Big)^2.$$

(i) If $D > 0$ and $f_{xx}(a,b) > 0$, then

$f(a,b)$ is a local min.

(ii) If $D > 0$ and $f_{xx}(a,b) < 0$, then

$f(a,b)$ is a local max.

(iii) If $D < 0$, then

there is a saddle point at (a,b).

(c) To find the absolute max/min, take the max/min of the critical points and the boundary.

Exercise 8

Let $f(x,y) = x^2y + 2x - 1$. Find the critical points.

$f_x(x,y) = 2xy + 2 = 0 \Rightarrow 2 = 0.$ Not possible. So, there are no critical points.

$f_y(x,y) = x^2 = 0 \Rightarrow x = 0$

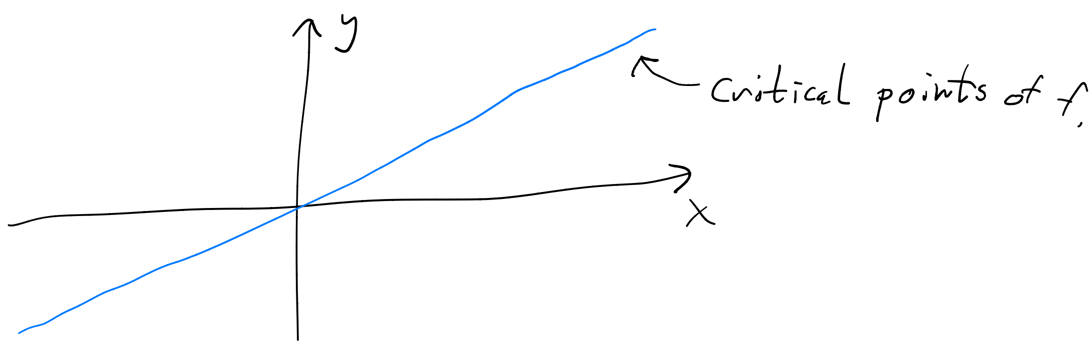
Exercise 9

Find all the critical points of $f(x, y) = x^2 - 4xy + 4y^2 + 2$. Show that $D = 0$ at every critical point. Show that each critical point is an absolute minimum of f.

$f_x(x,y) = 2x - 4y = 0 \Rightarrow y = \frac{1}{2}x$

$f_y(x,y) = -4x + 8y = 0 \Rightarrow y = \frac{1}{2}x$.

So, for any x, $(x, \frac{1}{2}x)$ is a critical point.



$f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 8$, $f_{xy}(x,y) = -4$

$D(x, \frac{1}{2}x) = f_{xx}(x, \frac{x}{2}) f_{yy}(x, \frac{x}{2}) - \left(f_{xy}(x, \frac{x}{2})\right)^2 = (2)(8) - (-4)^2 = 0$.

To show the critical points are absolute minima, note that $f(x,y) = (x-2y)^2 + 2$. From this, it is clear that the absolute minimum of f is 2. This occurs when $x - 2y = 0$. i.e., when $y = \frac{x}{2}$.

Exercise 10

Find all the critical points of $f(x, y) = (x - y)e^{-x^2-y^2}$.

$$f_x(x,y) = (x-y)(-2x)e^{-x^2-y^2} + e^{-x^2-y^2}$$
$$= [1-2x(x-y)]e^{-x^2-y^2} = 0$$

$$f_y(x,y) = (x-y)(-2y)e^{-x^2-y^2} - e^{-x^2-y^2}$$
$$= [-1-2y(x-y)]e^{-x^2-y^2} = 0$$

$$\Rightarrow 1 = 2x(x-y) \quad \text{and} \quad -1 = 2y(x-y)$$

$$\Rightarrow 1 = 2x^2 - 2xy$$
$$\Rightarrow 2xy = 2x^2 - 1$$
$$\Rightarrow y = x - \frac{1}{2x}$$

$$-1 = 2\left(x - \frac{1}{2x}\right)\left(\cancel{x} - \cancel{x} + \frac{1}{2x}\right)$$
$$-\cancel{2}x = \cancel{2}\left(x - \frac{1}{2x}\right)$$
$$\frac{1}{2x} = 2x \Rightarrow 4x^2 = 1 \Rightarrow x = \pm\frac{1}{2}$$

at $x = \frac{1}{2}$, $y = \frac{1}{2} - \frac{1}{2\left(\frac{1}{2}\right)} = -\frac{1}{2}$

at $x = \frac{-1}{2}$, $y = \frac{-1}{2} - \frac{1}{2\left(\frac{-1}{2}\right)} = \frac{1}{2}$

$\boxed{\text{critical points: } \left(\frac{-1}{2}, \frac{1}{2}\right) \text{ and } \left(\frac{1}{2}, \frac{-1}{2}\right).}$

Exercise 11

Show that $f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2$ has two local maxima, but no local minima. Notice that this is not possible for a function of one variable!

$$f_x(x,y) = -2(x^2-1)(2x) - 2(x^2y - x - 1)(2xy-1) = 0$$

$$= -4x^3 + 4x - 2(x^2y - x - 1)(2xy-1)$$

$$f_y(x,y) = -2(x^2y - x - 1)x^2 = 0$$

$x = 0$ or $x^2y - x - 1 = 0$

$\cancel{f_x(0,y) = -2 = 0}$

$f_x = -2(x^2-1)(2x) = 0$

$\Rightarrow x = 0$ or $x = -1$ or $x = 1$

$\cancel{-1 = 0}$

$y + 1 - 1 = 0 \Rightarrow y = 0$

$y - 1 - 1 = 0 \Rightarrow y = 2.$

critical points: $(-1,0)$ and $(1,2)$.

$$f_{xy}(x,y) = -2\left[(x^2y - x - 1)\cdot 2x + x^2(2xy-1)\right]$$

$$f_{xy}(-1,0) = -2\left[(0+1-1)\cdot 2(-1) + 1\cdot(0-1)\right] = 2$$

$$f_{xy}(1,2) = -2\left[(2-1-1)\cdot 2 + 1\cdot(4-1)\right] = -6$$

$f_{yy}(x,y) = -2x^4$ $\quad f_{yy}(-1,0) = -2$ $\quad f_{yy}(1,2) = -2$

$$f_{xx}(x,y) = -12x^2 + 4 - 2(2xy-1)(2xy-1) - 2(x^2y - x - 1)(2y)$$

$$f_{xx}(-1,0) = -12 + 4 - 2(-1)(-1) - 2(+1-1)(0) = -10$$

$$f_{xx}(1,2) = -12 + 4 - 2(4-1)(4-1) - 2(2-1-1)(4) = -26$$

$$D(-1,0) = (-10)(-2) - 2^2 = +16$$

and $f_{xx}(-1,0) < 0$, so there is a local max at $(-1,0)$.

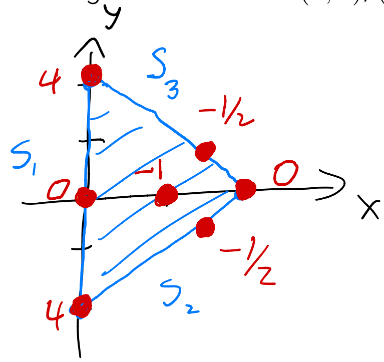
$$D(1,2) = (-26)(-2) - (-6)^2 = 52 - 36 = +16$$

and $f_{xx}(1,2) < 0$, so there is a local max at $(1,2)$.

There are no other critical points, so are no local minima.

Exercise 12

Find the absolute min and max of $f(x,y) = x^2 + y^2 - 2x$ over the triangle with vertices $(2,0)$, $(0,2)$, and $(0,-2)$.



$f_x(x,y) = 2x - 2 = 0 \Rightarrow x = 1$

$f_y(x,y) = 2y = 0 \Rightarrow y = 0$

critical point: $(1,0)$.

$f(1,0) = -1$

$\boxed{\text{absolute min: } -1 \quad \text{absolute max: } 4}$

Side S_1: $x = 0$, $-2 \le y \le 2$.

$f(0,y) = y^2$

Side S_2: $y = x - 2$, $0 \le x \le 2$.

$f(x, x-2) = x^2 + (x-2)^2 - 2x$

$= 2x^2 - 6x + 4$

$\frac{d}{dx} f(x, x-2) = 4x - 6 = 0 \Rightarrow x = \frac{3}{2}$

$f\left(\frac{3}{2}, \frac{3}{2} - 2\right) = \left(\frac{3}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 - 2\left(\frac{3}{2}\right) = \frac{-1}{2}$

Side S_3: $y = -x + 2$, $0 \le x \le 2$.

$f(x, 2-x) = x^2 + (2-x)^2 - 2x$

$= x^2 + 4 - 4x + x^2 - 2x$

$= 2x^2 - 6x + 4$

$\frac{d}{dx} f(x, 2-x) = 4x - 6 \Rightarrow x = \frac{3}{2}$

$f\left(\frac{3}{2}, 2 - \frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - 2\left(\frac{3}{2}\right)$

$= \frac{9}{4} + \frac{1}{4} - \frac{6}{2} = \frac{-2}{4} = \frac{-1}{2}$