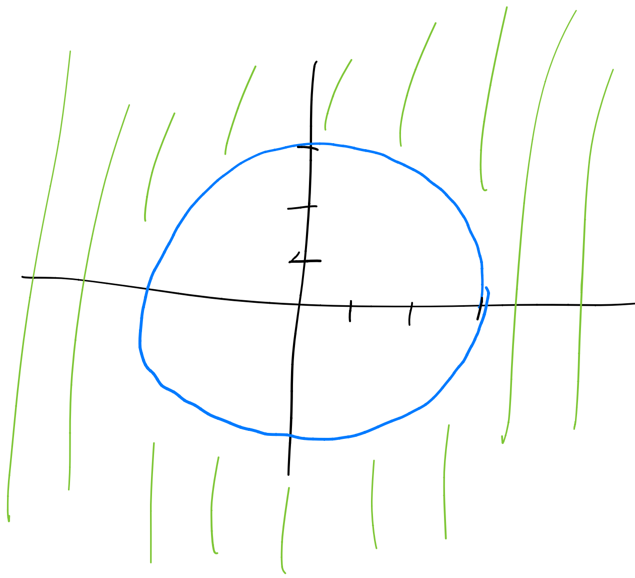
EXAM 2 REVIEW

Exercise 1

Find the domain and range of the following functions.

(a) $f(x, y) = \sqrt{x^2 + y^2 - 9}$

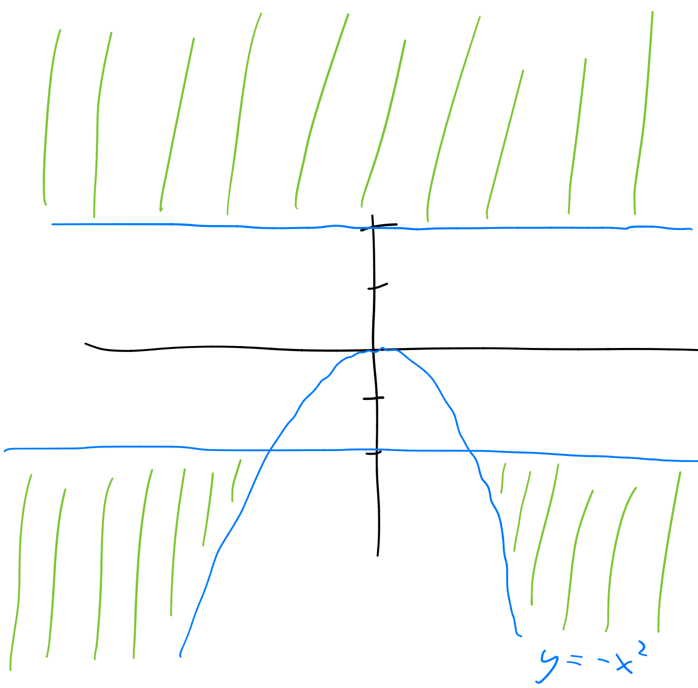
domain



$x^2 + y^2 - 9 \geq 0 \Rightarrow x^2 + y^2 \geq 9$

range: $[0, \infty)$.

(b) $g(x, y) = \sqrt{y^2 - 4} + \sqrt[4]{y + x^2}$



$y^2 - 4 \geq 0 \Rightarrow y^2 \geq 4$

$\Rightarrow y \geq 2$ or $y \leq -2$

$y + x^2 \geq 0 \Rightarrow y \geq -x^2$

range: $[0, \infty)$.

Exercise 2

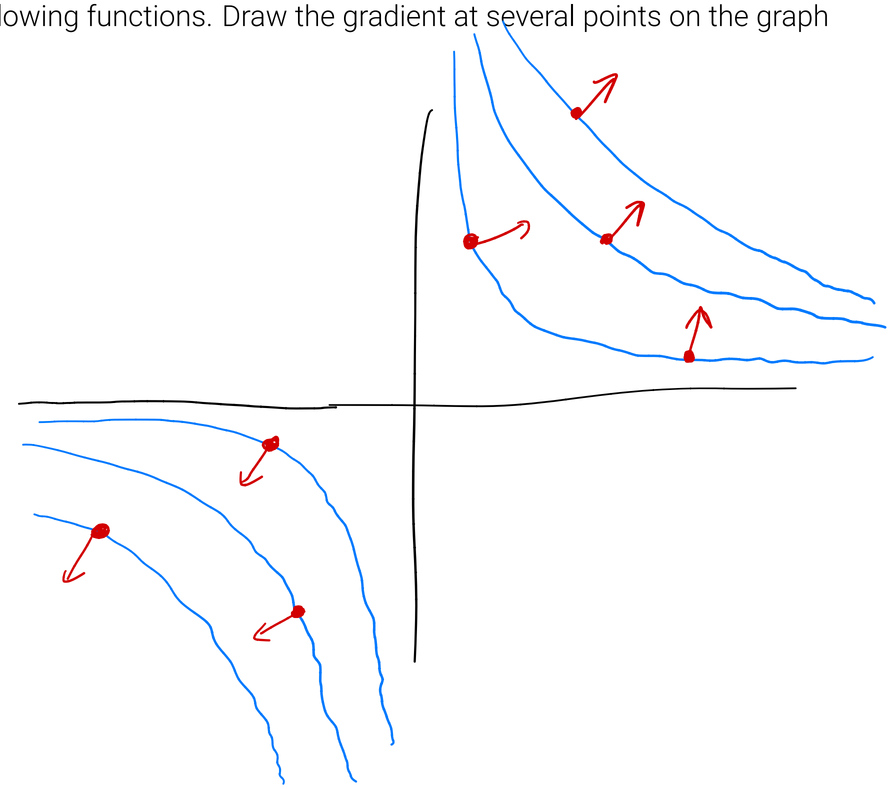
Draw some level curves of the following functions. Draw the gradient at several points on the graph of level curves.

(a) $f(x, y) = \ln(xy)$

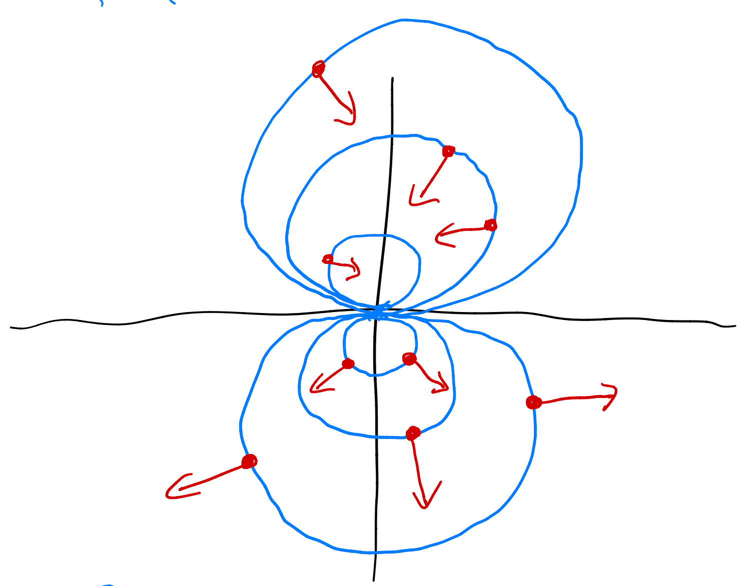
$\ln(xy) = c$

$xy = e^c$

$y = \frac{e^c}{x}$



(b) $g(x, y) = \frac{y}{x^2+y^2}$



$\frac{y}{x^2+y^2} = c$

$\frac{1}{c} y = x^2 + y^2$

$x^2 + y^2 - \frac{1}{c} y + \frac{1}{4c^2} =$

$x^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{4c^2} = \left(\frac{1}{2c}\right)^2$

center : $\left(0, \frac{1}{2c}\right)$ radius $= \frac{1}{2c}$

Exercise 3

The average energy E (in kcal) needed for a lizard to walk or run a distance of 1km has been modeled by the equation

$$E(m,v) = \frac{8}{3}m^{2/3} + \frac{7m^{3/4}}{2v},$$

where m is the body mass of the lizard (in grams) and v is the speed (in km/h). Compute $E_m(400,8)$ and $E_v(400,8)$ and interpret your results.

$$E_m(m,v) = \frac{16}{9}m^{-1/3} + \frac{21}{8}\frac{m^{-1/4}}{v}$$

$$E_m(400,8) = \frac{16}{9}400^{-1/3} + \frac{21}{8}\frac{800^{-1/4}}{8}$$

how much the required energy increases as the mass increases if the speed stays the same.
(heavier = more energy)

$$E_v(m,v) = \frac{-7m^{3/4}}{v^2}$$

$$E_v(m,v) = \frac{-7(400)^{3/4}}{64}$$

how much the required energy changes as the speed changes if weight stays same.
(faster = less energy)

Exercise 4

In a study of frost depth, it was found that the temperature T at time t (in days) at a depth x (in feet) can be modeled by the function

$$T(x,t) = T_0 + T_1 e^{-\lambda x}\sin(\frac{2\pi}{365}t - \lambda x),$$

where λ, T_0, and T_1 are some constants.

(a) Compute $\frac{\partial T}{\partial x}$. What is its physical significance?

$$\frac{\partial T}{\partial x} = -\lambda T_1 e^{-\lambda x}\sin\left(\frac{2\pi}{365}t - \lambda x\right) - \lambda T_1 e^{-\lambda x}\cos\left(\frac{2\pi}{365}t - \lambda x\right)$$

how fast the temperature changes as you change the depth while keeping the day the same.

(b) Compute $\frac{\partial T}{\partial t}$. What is its physical significance?

$$\frac{\partial T}{\partial t} = \frac{2\pi}{365}T_1 e^{-\lambda x}\cos\left(\frac{2\pi}{365}t - \lambda x\right)$$

how fast the temperature changes at a specific depth as time passes.

Exercise 5

Find the tangent plane to $z = \frac{x}{y^2}$ at the point $(-4, 2, -1)$.

$f_x(x,y) = \frac{1}{y^2}$ $\quad f_y(x,y) = \frac{-2x}{y^3}$

$f_x(-4,2) = \frac{1}{4}$ $\quad f_y(-4,2) = \frac{-2(-4)}{8} = 1$

$z = f(-4,2) + f_x(-4,2)(x+4) + f_y(-4,2)(y-2)$

$\boxed{z = -1 + \frac{1}{4}(x+4) + y - 2}$

Exercise 6

Use differentials to estimate the amount of metal in a closed cylindrical can that is 12cm high and 8cm in diameter if the tin is 0.04cm thick.

$V_{cylinder} = \pi r^2 h$

$dV = 2\pi r h\, dr + \pi r^2 dh$

$= 2\pi(4)(12)(0.04) + \pi(4)^2(0.08)$

Exercise 7

Use differentials to estimate the value of $\ln((1.1)^3 + (1.2)^2)$.

$$f(x,y) = \ln(x^3+y^2) \qquad f(1,1) = \ln(2)$$

$$df = \frac{3x^2}{x^3+y^2}\,dx + \frac{2y}{x^3+y^2}\,dy$$

$$= \frac{3}{2}(0.1) + \frac{2}{2}(0.2) = 0.15+0.2 = 0.35$$

$$f\left((1.1)^3+(1.2)^2\right) \approx \ln(2) + 0.35$$

$$\approx 0.693 + 0.35 = 1.04$$

Exercise 8

Recall the ideal gas law: $PV = nRT$. ($R = 8.31\,\frac{\text{kg}\cdot\text{m}^2}{\text{s}^2\cdot\text{K}\cdot\text{mol}}$) Suppose we have a closed box of 2 moles of a gas. If we increase the temperature according to $T(t) = (200 + t^2)$ K and change the volume according to $V(t) = (10 - t)\,\text{m}^3$, how fast is the pressure changing at time $t = 3$?

$$P = \frac{nRT}{V}$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial V}\frac{dV}{dt} + \frac{\partial P}{\partial T}\frac{dT}{dt}$$

$$= \frac{-nRT}{V^2}(-1) + \frac{nR}{V}(2t)$$

At $t=3$:

$V = 7\ m^3$

$T = 209$ K

$$\left.\frac{dP}{dt}\right|_{t=2} = \frac{+(2)(8.31)}{49} + \frac{(2)(8.31)}{7}(2)(3)$$

Exercise 9

The relative humidity can be expressed as the formula

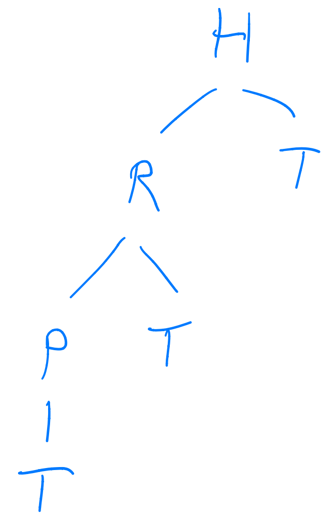
Assume ideal gas.

$$R(P,T,w) = \frac{w}{a + bPe^{\frac{T}{1+T}}},$$

$$P = \frac{nRT}{V}.$$

where P is the pressure, T is the temperature, and w is the amount of water in the air. The heat index is a function of R and T. Find an expression for how fast the heat index changes as the temperature is increased in a room of volume V with a fixed amount of water in the air.

$$\frac{dH}{dT} = \frac{\partial H}{\partial R}\left(\frac{\partial R}{\partial P}\frac{dP}{dt} + \frac{\partial R}{\partial T}\right) + \frac{\partial H}{\partial T}$$



Exercise 10

Compute the gradient of $f(x,y,z) = x^2\sin(yz) + 2y^2z$.

$$\nabla f(x,y,z) = \left(2x\sin(yz),\ x^2z\cos(yz) + 4yz,\ x^2y\cos(yz) + 2y^2\right)$$

Exercise 11

Compute the directional derivative of $g(x, y) = e^{xy^2}$ at the point $(0, 2)$ in the direction $\langle -4, 3\rangle$.

$$\nabla g(x,y) = \langle y^2 e^{xy^2}, 2xye^{xy^2} \rangle$$

$$\vec{u} = \frac{1}{\sqrt{(-4)^2+3^2}} \langle -4, 3 \rangle = \langle \frac{-4}{5}, \frac{3}{5} \rangle.$$

$$D_{\vec{u}} g(0,2) = \nabla g(0,2) \cdot \vec{u}$$

$$= \langle 4, 0 \rangle \cdot \langle \frac{-4}{5}, \frac{3}{5} \rangle$$

$$= \boxed{\frac{-16}{5}}$$

Exercise 12

At what point on the ellipsoid $x^2+y^2+2z^2 = 1$ is the tangent plane parallel to the plane $x+2y+z = 1$?

Think of $\underbrace{x^2+y^2+2z^2}_{g(x,y,z)}=1$ as a level surface. Then, ∇g is $\perp$ to this level surface.

$\vec{n} = \langle 1, 2, 1 \rangle.$

$$\nabla g(x,y,z) = \langle 2x, 2y, 4z \rangle = c \langle 1, 2, 1 \rangle$$

$\Rightarrow$ $2x = c$

$2y = 2c$

$4z = c$

$x^2+y^2+2z^2=1$

$x = 2z$

$y = 4z$

$(2z)^2 + (4z)^2 + 2z^2 = 1$

$22z^2 = 1$

$\Rightarrow z = \pm \frac{1}{\sqrt{22}}$

$$\boxed{\left(\frac{-2}{\sqrt{22}}, \frac{-4}{\sqrt{22}}, \frac{-1}{\sqrt{22}}\right) \text{ and } \left(\frac{2}{\sqrt{22}}, \frac{4}{\sqrt{22}}, \frac{1}{\sqrt{22}}\right)}$$

Exercise 13

The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2},$$

where T is measured in °C and x, y, z are measured in meters.

(a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $Q(3, -3, 3)$.

$$\overrightarrow{PQ} = \langle 1, -2, 1 \rangle$$

$$\vec{u} = \frac{1}{|\overrightarrow{PQ}|}\overrightarrow{PQ} = \left\langle \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$\nabla T(x,y,z) = \langle -400xe^{-x^2-3y^2-9z^2}, -1200ye^{-x^2-3y^2-9z^2}, -3600ze^{-x^2-3y^2-9z^2} \rangle$$

$$\nabla T(2,-1,2) = \langle -800e^{-43}, 1200e^{-43}, -7200e^{-43} \rangle$$

$$D_{\vec{u}} T(2,-1,2) = \frac{e^{-43}}{\sqrt{6}}(-800 - 2400 - 7200).$$

(b) In which direction does the temperature increase fastest at P?

in the direction of the gradient:

$$\langle -800e^{-43}, 1200e^{-43}, -7200e^{-43} \rangle$$

(c) Find the maximum rate of increase at P.

magnitude of the gradient:

$$\sqrt{(-800e^{-43})^2 + (1200e^{-43})^2 + (-7200e^{-43})^2}$$

Exercise 14

Find the shortest distance from $(2, 0, -3)$ to the plane $\overbrace{x + y + z}^{g(x,y,z)} = 1$.

distance squared from (x,y,z) to $(2,0,-3)$ is

$$f(x,y,z) = (x-2)^2 + y^2 + (x+3)^2.$$

$$\nabla f = \lambda \nabla g$$

$$\langle 2(x-2), 2y, 2(z+3) \rangle = \lambda \langle 1,1,1 \rangle$$

$$\boxed{\begin{aligned} 2x - 4 &= \lambda \\ 2y &= \lambda \\ 2z + 6 &= \lambda \\ x + y + z &= 1 \end{aligned}}$$

solve these for x, y, z.

$$2x - 4 = 2y \Rightarrow x = y + 2$$

$$2z + 6 = 2y \Rightarrow z = y - 3$$

$$y + 2 + y + y - 3 = 1$$

$$\Rightarrow 3y = 2 \Rightarrow y = \tfrac{2}{3} \Rightarrow x = 2 + \tfrac{2}{3}$$

$$z = \tfrac{2}{3} - 3$$

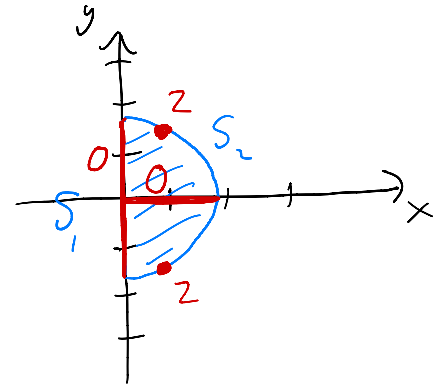
shortest distance: $\sqrt{\left(\left(2+\frac{2}{3}\right)-2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\left(\frac{2}{3}-3\right)+3\right)^2}$

$$= \sqrt{3\left(\tfrac{4}{9}\right)} = \boxed{\frac{2}{\sqrt{3}}}$$

Exercise 15

Find the absolute min and max of $g(x, y) = xy^2$ on the region $R = \{(x, y) : x \geq 0, x^2 + y^2 \leq 3\}$.

look for critical points:



$g_x(x,y) = y^2 = 0 \Rightarrow y = 0$

$g_y(x,y) = 2xy = 0 \Rightarrow x$ is anything

$g(x,0) = 0.$

Side S_1: $x = 0, \ -\sqrt{3} \leq y \leq \sqrt{3}.$

$g(0,y) = 0.$

Side S_2: $x = \sqrt{3-y^2}$

abs min: 0

abs max: 2

$g(\sqrt{3-y^2}, y) = \sqrt{3-y^2}\, y^2$

$\frac{d}{dy} g(\sqrt{3-y^2}, y) = 2y\sqrt{3-y^2} - y\frac{y^2}{\sqrt{3-y^2}} = 0$

$\Rightarrow 2\sqrt{3-y^2} = \frac{y^2}{\sqrt{3-y^2}}$

$\Rightarrow 2(3-y^2) = y^2$

$\Rightarrow 6 - 2y^2 = y^2 \Rightarrow y^2 = 2$

$\Rightarrow y = \pm\sqrt{2}$

$g(\sqrt{3-(\sqrt{2})^2}, -\sqrt{2}) = 2$

$g(\sqrt{3-(\sqrt{2})^2}, \sqrt{2}) = 2$

Exercise 16

Using Lagrange multipliers, find the max and min of $f(x, y) = 2x + 2y + z$ subject to the constraint $\underbrace{x^2 + y^2 + z^2}_{g(x,y,z)} = 9$.

$\nabla f = \lambda \nabla g$

$< 2, 2, 1 > = \lambda < 2x, 2y, 2z >$

$2 = 2\lambda x$

$2 = 2\lambda y$

$1 = 2\lambda z$

$x^2 + y^2 + z^2 = 9$

Solve for x, y, z.

$\Rightarrow x = y$

$\Rightarrow 2z = x \Rightarrow y = 2z$

$\Rightarrow (2z)^2 + (2z)^2 + z^2 = 9$

$9z^2 = 9$

$\Rightarrow z = \pm 1$

$z = -1$: $x = -2$, $y = -2$

$z = 1$: $x = 2$, $y = 2$

$f(-2, -2, -1) = 2(-2) + 2(-2) - 1 = -9$

$f(2, 2, 1) = 2(2) + 2(2) + 1 = 9$

$\boxed{\text{max} : 9 \quad \text{min} : -9}$