15.1 — DOUBLE INTEGRALS OVER RECTANGLES

Review

(a) We can take double integrals by using an iterated integral. If $D = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, then
$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \int_a^b f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

(b) The order of integration can make the problem much easier or harder.

(c) The double integral of f gives the volume under the surface $f(x, y)$.

Exercise 1

Compute the following double integrals.

(a) $\displaystyle\int_0^1 \int_0^1 (x+y)^2\,\mathrm{d}x\,\mathrm{d}y$

$$= \int_0^1 \int_0^1 (x^2 + 2xy + y^2)\,dx\,dy$$

$$= \int_0^1 \left(\frac{1}{3}x^3 + x^2y + xy^2\right)\Big|_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{1}{3} + y + y^2\right) dy$$

$$= \frac{1}{3}y + \frac{1}{2}y^2 + \frac{1}{3}y^3 \Big|_0^1$$

$$= \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \boxed{\frac{7}{6}}$$

Exercise 1 continued on next page...

(b) $\displaystyle\int_1^3 \int_1^5 \frac{\ln(y)}{xy}\,dy\,dx$

$u = \ln(y)$

$du = \frac{1}{y}dy$

$$= \int_1^3 \int_0^{\ln(5)} \frac{u}{x}\,du\,dx$$

$$= \int_1^3 \frac{1}{2x}\, u^2 \Big|_{u=0}^{u=\ln(5)} dx$$

$$= \int_1^3 \frac{1}{2x} \ln^2(5)\,dx$$

$$= \frac{\ln^2(5)}{2} \ln(x)\Big|_{x=1}^{x=3} = \boxed{\frac{\ln^2(5)\ln(3)}{2}}$$

(c) $\displaystyle\iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\,dA$, where $R = \{(\theta, t) : 0 \le \theta \le \pi/3, 0 \le t \le \frac{1}{2}\}$.

$$= \int_0^{\pi/3} \int_0^{1/2} \frac{\tan\theta}{\sqrt{1-t^2}}\,dt\,d\theta$$

$$= \int_0^{\pi/3} \tan\theta \ \arcsin(t)\Big|_{t=0}^{t=1/2} d\theta$$

$$= \int_0^{\pi/3} \tan\theta \left(\arcsin\left(\tfrac{1}{2}\right) - \arcsin(0)\right) d\theta$$

$$= \int_0^{\pi/3} \tan\theta \left(\frac{\pi}{6} - 0\right) d\theta$$

$$= \frac{\pi}{6} \ln|\sec\theta|\Big|_{\theta=0}^{\theta=\pi/3}$$

$$= \frac{\pi}{6}\left(\ln|2| - \ln|1|\right) = \boxed{\frac{\pi}{6}\ln(2)}$$

$$\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$= \int \frac{-1}{u}\,du$$

$$= -\ln(|u|)$$

$$= -\ln(|\cos\theta|)$$

$$= \ln(|\sec\theta|).$$

Exercise 2

Find the volume of a solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

$$\int_{-1}^{1}\int_{1}^{2}(3y^2 - x^2 + 2)\,dy\,dx$$

$$= \int_{-1}^{1}\left(y^3 - x^2 y + 2y\right)\Big|_{y=1}^{y=2} dx$$

$$= \int_{-1}^{1}\left(8 - 2x^2 + 4 - 1 + x^2 - 2\right) dx$$

$$= \int_{-1}^{1}(9 - x^2)\,dx$$

$$= 9x - \frac{1}{3}x^3\Big|_{x=-1}^{x=1}$$

$$= 9 - \frac{1}{3} - \left(-9 - \frac{1}{3}(-1)\right) = 9 - \frac{1}{3} + 9 - \frac{1}{3} = \boxed{18 - \frac{2}{3}}$$

Exercise 3

Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

$$\int_{0}^{4}\int_{0}^{5}(16 - x^2)\,dy\,dx$$

where does $z = 16 - x^2$ intersect $z = 0$?

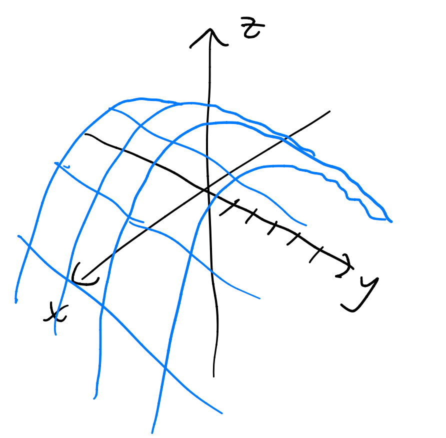
$0 = 16 - x^2$

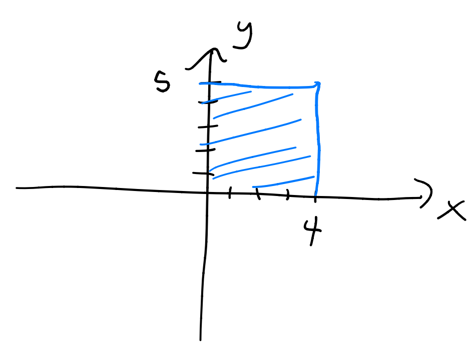
$\Rightarrow x = \pm 4$

$$= \int_{0}^{4} 5(16 - x^2)\,dx$$

$$= 5\left(16x - \frac{1}{3}x^3\right)\Big|_{x=0}^{4}$$

$$= \boxed{5\left(64 - \frac{64}{3}\right)}$$





15.2 — DOUBLE INTEGRALS OVER GENERAL REGIONS

Review

(a) If $D = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$, then

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

(b) If $D = \{(x,y) : g_1(y) \le x \le g_2(y), a \le y \le b\}$, then

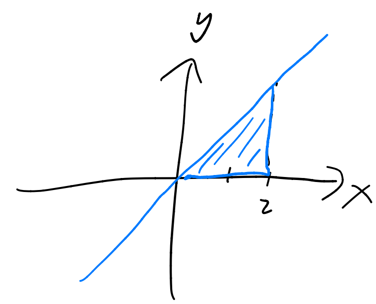
$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

(c) The order of integration can make the problem much easier or harder.

Exercise 4

Evaluate the following double integrals.

(a) $\iint_D y\sqrt{x^2 - y^2}\,\mathrm{d}A$, where $D = \{(x,y) : 0 \le x \le 2, 0 \le y \le x\}$.



$$\int_0^2 \int_0^x y\sqrt{x^2-y^2}\,dy\,dx$$

$u = x^2 - y^2$

$du = -2y\,dy$

$$= \int_0^2 -\frac{1}{2}\int_{x^2}^0 u^{1/2}\,du\,dx$$

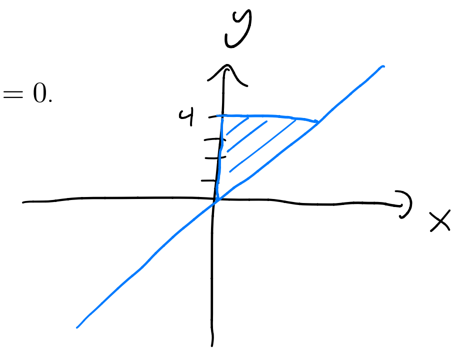
$$= \int_0^2 -\frac{1}{2}\,\frac{2}{3}\,u^{3/2}\Big|_{u=x^2}^{u=0}\,dx$$

$$= +\frac{1}{3}\int_0^2 x^3\,dx$$

$$= \frac{1}{12}\,x^4\Big|_0^2 = \frac{16}{12} = \boxed{\frac{4}{3}}$$

Exercise 4 continued on next page…

(b) $\iint_D y^2 e^{xy}\,\mathrm{d}A$, where D is the region bounded by $y = x$, $y = 4$, and $x = 0$.



$$\int_0^4 \int_0^y y^2 e^{xy}\,dx\,dy$$

$$= \int_0^4 y e^{xy}\Big|_{x=0}^{x=y} dy$$

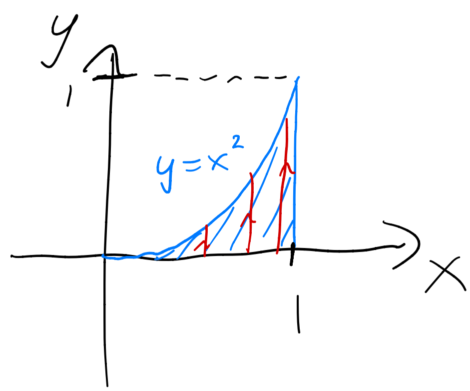
$$= \int_0^4 \left(y e^{y^2} - y\right) dy$$

$$= \int_0^4 y e^{y^2} dy - \frac{1}{2} y^2 \Big|_0^4$$

$u = y^2$
$du = 2y$

$$= \frac{1}{2}\int_0^{16} e^u\,du - \frac{1}{2}(16-0) = \boxed{\frac{1}{2}\left(e^{16}-1\right) - 8}$$

(c) $\iint_D x\cos(y)\,\mathrm{d}A$, where D is bounded by $y = 0$, $y = x^2$, and $x = 1$.



$$\int_0^1 \int_0^{x^2} x\cos(y)\,dy\,dx$$

$$= \int_0^1 x \sin(y)\Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 x\sin(x^2)\,dx$$

$u = x^2$
$du = 2x\,dx$

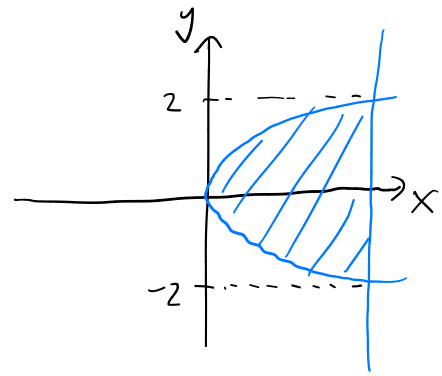
$$= \frac{1}{2}\int_0^1 \sin(u)\,du$$

$$= \boxed{\frac{1}{2}\sin(1)}$$

Exercise 5

Find the volume under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$.

$$\int_{-2}^{2}\int_{y^2}^{4}(1+x^2y^2)\,dx\,dy$$



$$= \int_{-2}^{2}\left(x + \frac{1}{3}x^3y^2\right)\Big|_{x=y^2}^{x=4}\,dy$$

$$= \int_{-2}^{2}\left(4 + \frac{64}{3}y^2 - y^2 - \frac{1}{3}y^8\right)dy$$

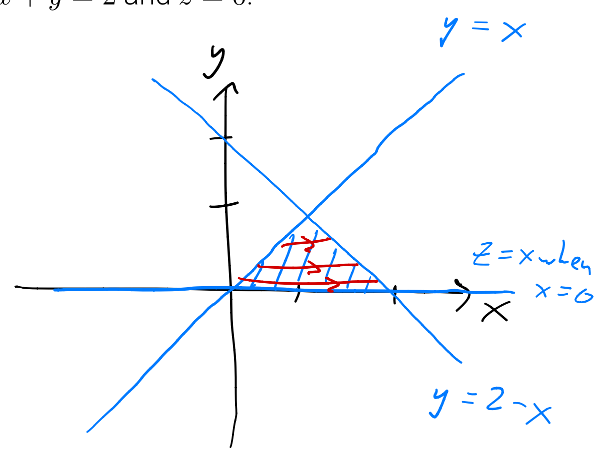
$$= \int_{-2}^{2}\left(4 + \frac{61}{3}y^2 - \frac{1}{3}y^8\right)dy = 4y + \frac{61}{6}y^3 - \frac{1}{27}y^9$$

$$= 4(2) + \frac{61}{6}(8) - \frac{1}{27}2^9 - \left(4(-2) + \frac{61}{6}(-2)^3 - \frac{1}{27}(-2)^9\right)$$

Exercise 6

Find the volume of the region bounded by the planes $z = x$, $y = x$, $x + y = 2$ and $z = 0$.

$$\int_0^1\int_y^{2-y} x\,dx\,dy = \int_0^1 \frac{1}{2}x^2\Big|_{x=y}^{x=2-y}\,dy$$



$$= \int_0^1 \frac{1}{2}\left((2-y)^2 - y^2\right)dy$$

$$= \int_0^1 \frac{1}{2}\left(4 - 4y + \cancel{y^2} - \cancel{y^2}\right)dy$$

$$= \int_0^1 (2 - 2y)\,dy$$

$$= 2y - y^2\Big|_{y=0}^{1} = 2 - 1 = \boxed{1}$$

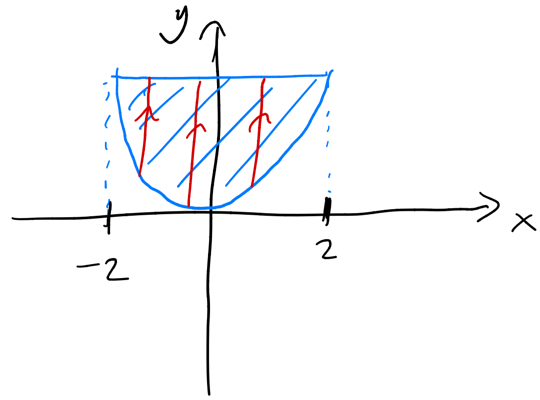
Exercise 7

Find the volume of the region bounded by the cylinders $z = x^2$ and $y = x^2$ and the planes $z = 0$ and $y = 4$.

$$\int_{-2}^{2}\int_{x^2}^{4} x^2\,dy\,dx = \int_{-2}^{2}(4x^2 - x^4)\,dx$$

$$= \frac{4}{3}x^3 - \frac{1}{5}x^5 \Big|_{x=-2}^{x=2}$$

$$= \frac{4}{3}(8) - \frac{32}{5} - \left(\frac{4}{3}(-2)^3 - \frac{1}{5}(-2)^5\right).$$

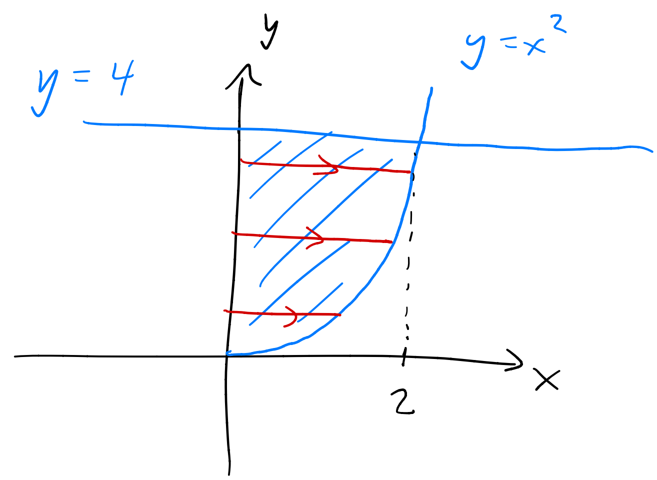


Exercise 8

Switch the order of integration in the following.

(a) $\int_0^2 \int_{x^2}^4 f(x,y)\,\mathrm{d}y\,\mathrm{d}x$

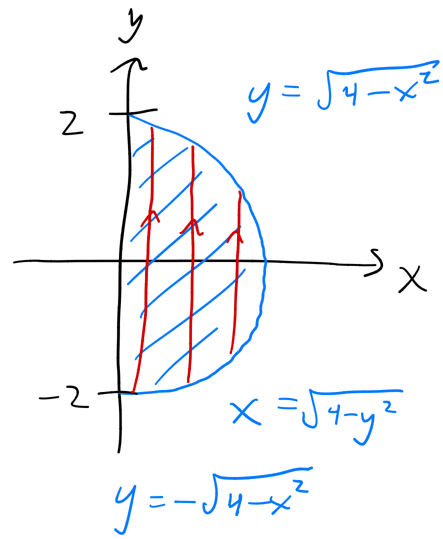
$$\int_0^4 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$$



Exercise 8 continued on next page. . .

(b) $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} \mathrm{d}x\,\mathrm{d}y$

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx$$

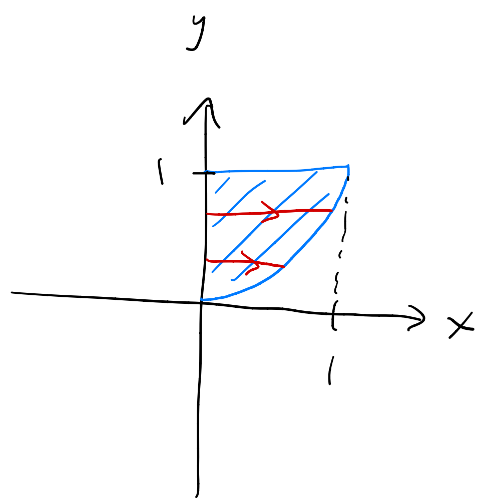


Exercise 9

Evaluate the following.

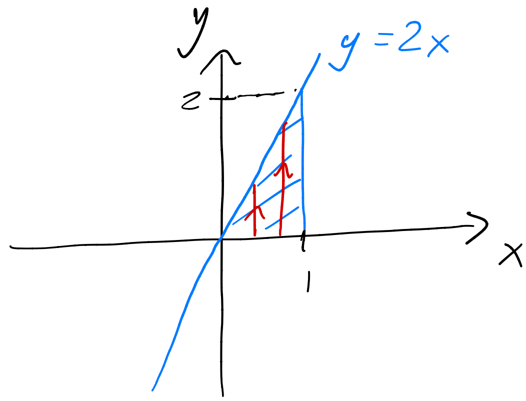
(a) $\displaystyle\int_{0}^{1}\int_{x^2}^{1} \sqrt{y}\sin(y)\,\mathrm{d}y\,\mathrm{d}x$

$$\int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\,\sin(y)\,dx\,dy$$



Exercise 9 continued on next page. . .

(b) $\int_0^2 \int_{y/2}^1 y\cos(x^3-1)\,\mathrm{d}x\,\mathrm{d}y$



$$\int_0^1 \int_0^{2x} y\cos(x^3-1)\,dy\,dx$$

$$= \int_0^1 \frac{1}{2}y^2\cos(x^3-1)\Big|_{y=0}^{y=2x}\,dx$$

$$= \int_0^1 2x^2\cos(x^3-1)\,dx \qquad u = x^3-1, \quad du = 3x^2\,dx$$

$$= \frac{2}{3}\int_{-1}^0 \cos(u)\,du = \frac{2}{3}\left[\sin(0)-\sin(-1)\right] = \boxed{-\frac{2}{3}\sin(-1)}$$

15.3 — DOUBLE INTEGRALS IN POLAR COORDS

Review

(a) If $D = \{(r,\theta) : a \le r \le b, c \le \theta \le d\}$, then

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_c^d f(r\cos\theta, r\sin\theta)\,r\,\mathrm{d}\theta\,\mathrm{d}r.$$

(b) If $D = \{(r,\theta) : g_1(\theta) \le r \le g_2(\theta), a \le r \le b\}$, then

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta.$$

(c) If $D = \{(r,\theta) : a \le r \le b, g_1(r) \le \theta \le g_2(r)\}$, then

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)\,r\,\mathrm{d}\theta\,\mathrm{d}r.$$

(d) The order of integration can make the problem much easier or harder.

Exercise 10

Find the volume ~~under~~ between the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant.

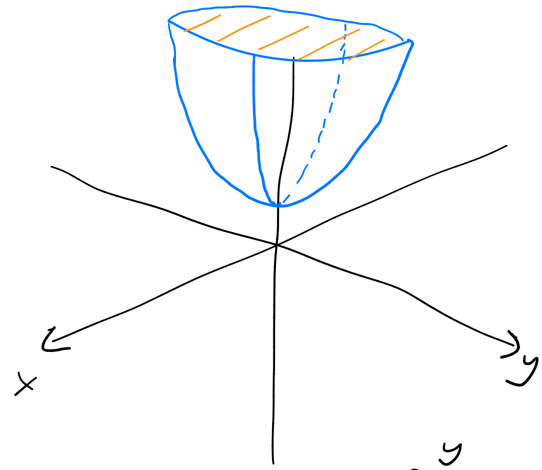
$$\int_0^{\pi/4}\int_0^{\sqrt{3}} \left(7-(1+2r^2)\right) r\,dr\,d\theta$$

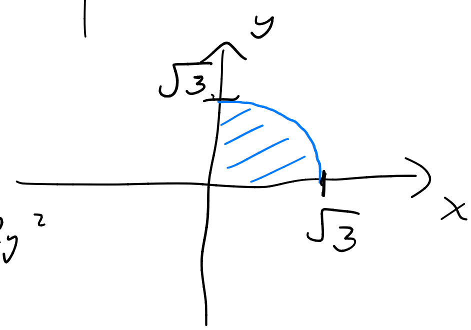
$$= \int_0^{\pi/4}\int_0^{\sqrt{3}} (6r - 2r^3)\,dr\,d\theta$$

$$= \int_0^{\pi/4} 3r^2 - \frac{2}{4}r^4 \Big|_{r=0}^{\sqrt{3}}\,d\theta$$

$$= \int_0^{\pi/4} \left(3(3) - \frac{1}{2}(9)\right) d\theta$$

$$= \frac{\pi}{4}\left(\frac{9}{2}\right) = \boxed{\frac{9\pi}{8}}$$





$7 = 1 + 2x^2 + 2y^2$

$x^2 + y^2 = 3$

Exercise 11

Find the volume above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

intersection:

$z = \sqrt{1 - x^2 - y^2}$

$\sqrt{x^2+y^2} = \sqrt{1-x^2-y^2}$

$x^2 + y^2 = 1 - x^2 - y^2$

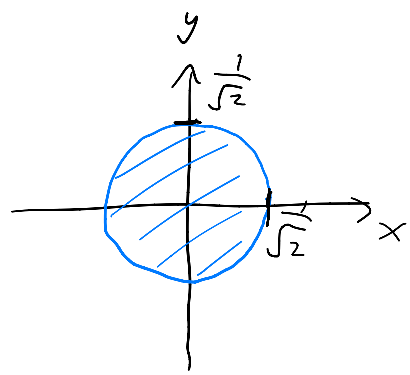
$x^2 + y^2 = \frac{1}{2}$

$$\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1-r^2} - r\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int_0^{\frac{1}{\sqrt{2}}} r\sqrt{1-r^2}\,dr - \frac{1}{3}r^3\Big|_{r=0}^{\frac{1}{\sqrt{2}}}\right] d\theta$$

$u = 1 - r^2$

$du = -2r\,dr$



$$= \int_0^{2\pi}\left[-\frac{1}{2}\int_1^{1/2} u^{1/2}\,du - \frac{1}{6\sqrt{2}}\right] d\theta$$

$$= \int_0^{2\pi}\left[-\frac{2}{3}\left(\frac{1}{2}\right)u^{3/2}\Big|_{u=1}^{1/2} - \frac{1}{6\sqrt{2}}\right] d\theta$$

$$= 2\pi\left(-\frac{1}{3}\left(\frac{1}{2^{3/2}} - 1\right) - \frac{1}{6\sqrt{2}}\right).$$

Exercise 12

Find the volume bounded by the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$.

$$\int_0^{2\pi}\int_0^{\sqrt{2}} (6 - r^2 - 2r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} (6r - 3r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(3r^2 - \frac{3}{4}r^4\right)\Big|_{r=0}^{r=\sqrt{2}} d\theta$$

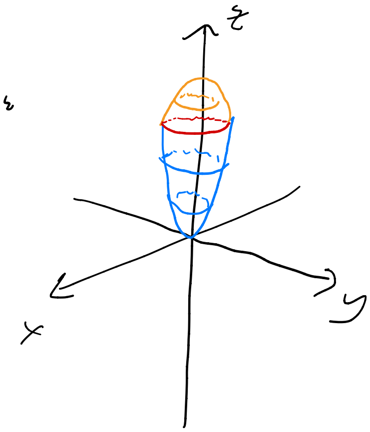
$$= 2\pi\left(3(2) - \frac{3}{4}(4)\right)$$

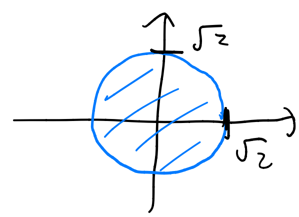
$$= \boxed{6\pi}$$

intersection:

$$6 - x^2 - y^2 = 2x^2 + 2y^2$$

$$x^2 + y^2 = 2$$





Exercise 13

Compute $\displaystyle\int_0^2 \int_0^{\sqrt{2x - x^2}} \sqrt{x^2 + y^2}\, dy\, dx.$

$$y = \sqrt{2x - x^2}$$

$$y^2 = 2x - x^2 \rightarrow r^2 = 2r\cos\theta$$

$$\Rightarrow r = 2\cos\theta$$

$$x^2 - 2x + 1 + y^2 = 1$$

$$(x-1)^2 + y^2 = 1$$

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\, r\, dr\, d\theta$$

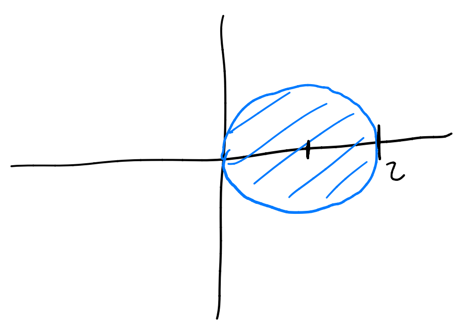
$$= \int_{-\pi/2}^{\pi/2} \frac{1}{3} r^3 \Big|_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{3}(2\cos\theta)^3 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{8}{3}(1 - \sin^2\theta)\cos\theta\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$



$$= \int_{-1}^{1} \frac{8}{3}(1 - u^2)\, du = \frac{8}{3}\left(u - \frac{1}{3}u^3\right)\Big|_{u=-1}^{u=1} \longrightarrow = \frac{8}{3}\left(1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right) = \boxed{\frac{32}{9}}$$