15.6 — TRIPLE INTEGRALS

Review

(a) We can take triple integrals by using an iterated integral. If $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iint_D f(x, y)\,\mathrm{d}A = \int_a^b \int_c^d \int_r^s f(x, y)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

(b) The order of integration can make the problem much easier or harder.

(c) The triple integral of 1 gives you the volume of the region of integration.

Exercise 1

Compute the following triple integrals.

(a) $\int_0^3 \int_0^x \int_{x-y}^{x+y} y\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$

$$= \int_0^3 \int_0^x yz \Big|_{z=x-y}^{z=x+y} dy\,dx$$

$$= \int_0^3 \int_0^x \big(y(x+y) - y(x-y)\big)\,dy\,dx$$

$$= \int_0^3 \int_0^x 2y^2\,dy\,dx$$

$$= \int_0^3 \frac{2}{3} y^3 \Big|_{y=0}^{y=x} dx$$

$$= \int_0^3 \frac{2}{3} x^3\,dx$$

$$= \frac{2}{12} x^4 \Big|_0^3 = \frac{2}{12} 3^4 = \boxed{\frac{27}{2}}$$

Exercise 1 continued on next page…

(b) $\int_0^1 \int_y^1 \int_0^{xy} e^{z/y}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$

$$= \int_0^1 \int_y^1 y e^{\frac{z}{y}} \Big|_{z=0}^{z=xy} dx\,dy$$

$$= \int_0^1 \int_y^1 (ye^x - y)\,dx\,dy$$

$$= \int_0^1 (ye^x - xy)\Big|_{x=y}^{x=1} dy$$

$$= \int_0^1 (ye - y - ye^y + y^2)\,dy$$

$$= \int_0^1 [(e-1)y + y^2]\,dy - \int_0^1 ye^y\,dy \qquad \begin{array}{ll} u = y & dv = e^y dy \\ du = dy & v = e^y \end{array}$$

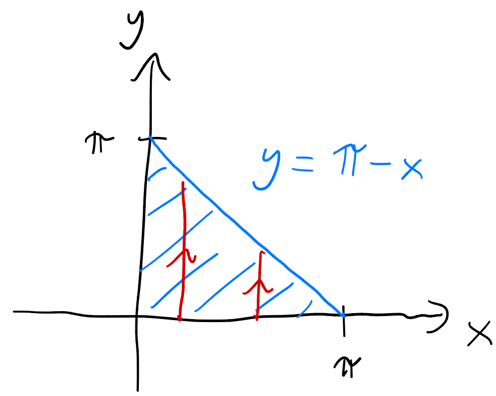
$$= \frac{e-1}{2}y^2 + \frac{1}{3}y^3 \Big|_{y=0}^{1} - ye^y\Big|_0^1 + \int_0^1 e^y\,dy$$

$$= \frac{e}{2} - \frac{1}{2} + \frac{1}{3} - e + e^y\Big|_0^1$$

$$= \frac{-1}{2}e - \frac{1}{6} + e - 1 = \boxed{\frac{1}{2}e - \frac{7}{6}}$$

Exercise 1 continued on next page...

(c) $\iiint_E \sin(y)\,\mathrm{d}V$, where E is the region below the plane $z=x$ and above the triangular region with vertices $(0,0,0)$, $(\pi,0,0)$, and $(0,\pi,0)$.



$$\int_0^{\pi}\int_0^{\pi-x}\int_0^{x} \sin(y)\,dz\,dy\,dx$$

$$= \int_0^{\pi}\int_0^{\pi-x} z\sin(y)\Big|_{z=0}^{z=x}\,dy\,dx$$

$$= \int_0^{\pi}\int_0^{\pi-x} x\sin(y)\,dy\,dx$$

$$= \int_0^{\pi} -x\cos(y)\Big|_{y=0}^{y=\pi-x}\,dx$$

$$= -\int_0^{\pi}\left(x\cos(\pi-x)-x\right)dx$$

$u=\pi-x \Rightarrow x=\pi-u$

$du=-dx$

$$= \int_{\pi}^{0}(\pi-u)\cos(u)\,du + \int_0^{\pi} x\,dx$$

$$= \int_{\pi}^{0}\pi\cos(u)\,du - \int_{\pi}^{0} u\cos(u)\,du + \frac{1}{2}x^2\Big|_0^{\pi}$$

$u_1=u \quad dv=\cos(u)du$

$du_1=du \quad v=\sin(u)$

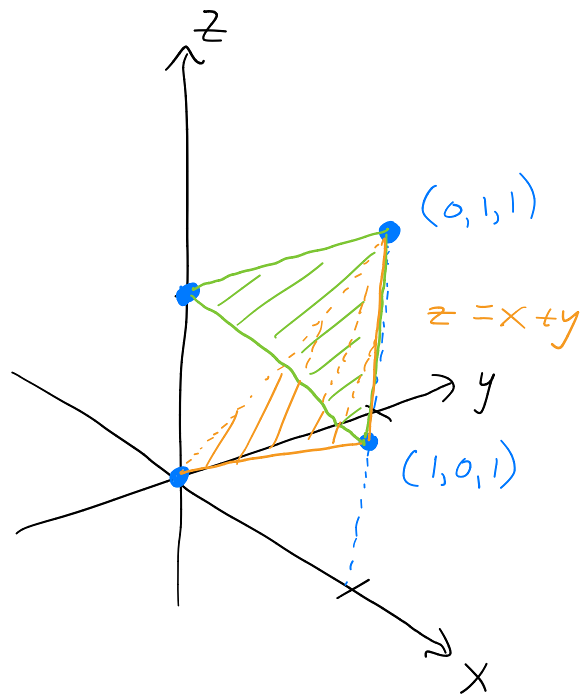
$$= \cancelto{0}{\pi\sin(u)\Big|_{\pi}^{0}} - \cancelto{0}{u\sin(u)\Big|_{\pi}^{0}} + \int_{\pi}^{0}\sin(u)\,du + \frac{1}{2}\pi^2 = -\cos(u)\Big|_{\pi}^{0} + \frac{1}{2}\pi^2$$

$$= -\cos(0)+\cos(\pi)+\frac{1}{2}\pi^2$$

$$= \boxed{\frac{1}{2}\pi^2 - 2}$$

Exercise 1 continued on next page...

(d) $\iiint_E xz\,\mathrm{d}V$, where E is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,1)$, $(0,1,1)$, and $(0,0,1)$.



Find the orange plane:

$\langle 1,0,1\rangle$ and $\langle 0,1,1\rangle$ are vectors in the plane. So, a normal vector to the plane is

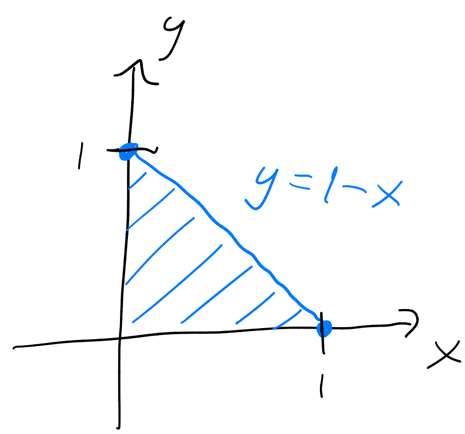
$$\vec{n} = \langle 1,0,1\rangle \times \langle 0,1,1\rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \langle -1,-1,1\rangle$$

equation for the orange plane:

$$-(x-0)-(y-0)+(z-0)=0$$

$$z = x+y$$



$$\int_0^1 \int_0^{1-x} \int_{x+y}^1 xz\,dz\,dy\,dx = \cdots = \boxed{\frac{1}{30}}$$

Exercise 2

Find the volume of the solid enclosed by the paraboloids $x = y^2 + z^2$ and $x = 4 - y^2 - z^2$.

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} 1 \, r \, dx \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} (4 - r^2 - r^2) r \, dr \, d\theta$$

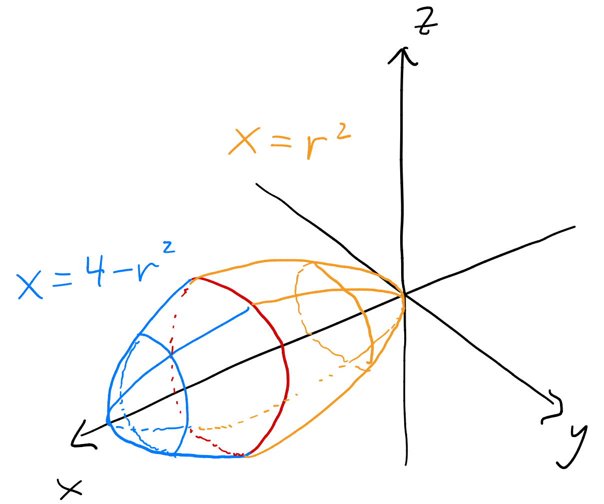
$$= \int_0^{2\pi} \int_0^{\sqrt{2}} (4r - 2r^3) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{4}{2} r^2 - \frac{2}{4} r^4 \right) \Bigg|_{r=0}^{r=\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \left(2(2) - \tfrac{1}{2}(4) \right) d\theta$$

$$= \int_0^{2\pi} 2 \, d\theta$$

$$= \boxed{4\pi}$$



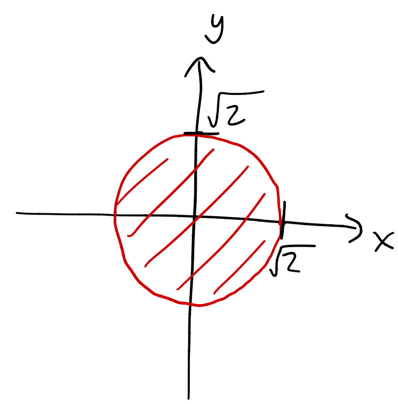
intersection:

$$4 - r^2 = r^2$$

$$\Rightarrow 2r^2 = 4$$

$$\Rightarrow r^2 = 2$$

$$\Rightarrow r = \sqrt{2}$$



Exercise 3

Find the volume of the solid enclosed by the cylinder $x^2+z^2=4$ and the planes $y=-1$ and $y+z=4$.

$$\int_0^{2\pi}\int_0^{2}\int_{-1}^{4-r\sin\theta} 1\, r\, dy\, dr\, d\theta$$

$$=\int_0^{2\pi}\int_0^{2} ry\Big|_{y=-1}^{r=4-r\sin\theta} dr\, d\theta$$

$$=\int_0^{2\pi}\int_0^{2} r\left[4-r\sin\theta+1\right] dr\, d\theta$$

$$=\int_0^{2\pi}\int_0^{2} \left(5r-r^2\sin\theta\right) dr\, d\theta$$

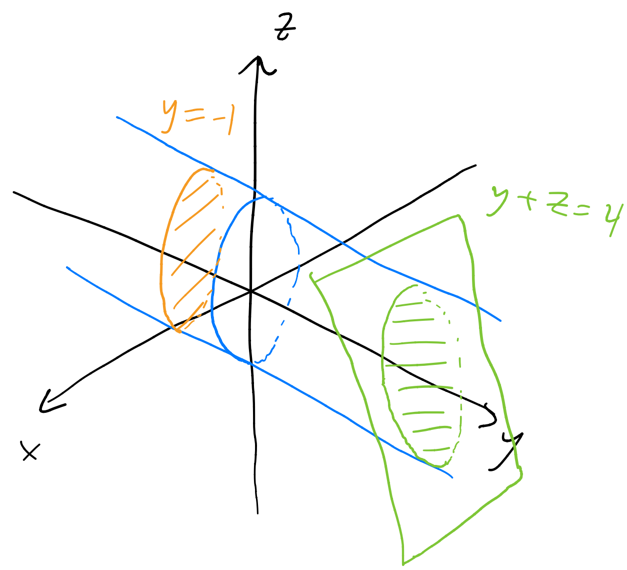
$$=\int_0^{2\pi} \left(\frac{5}{2}r^2-\frac{1}{3}r^3\sin\theta\right)\Big|_{r=0}^{r=2} d\theta$$

$$=\int_0^{2\pi} \left(\frac{5}{2}(4)-\frac{8}{3}\sin\theta\right) d\theta$$

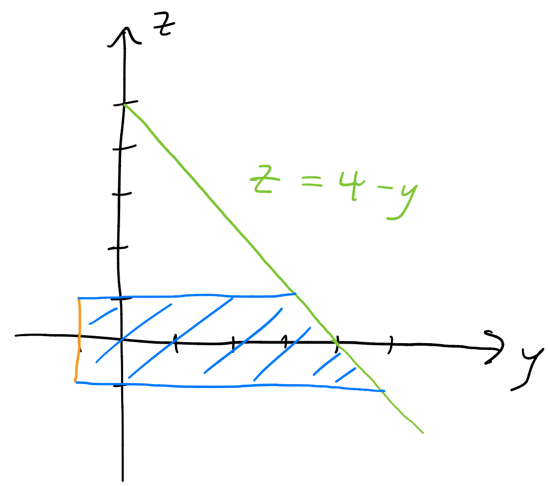
$$=\int_0^{2\pi} \left(10-\frac{8}{3}\sin\theta\right) d\theta$$

$$=10\theta+\frac{8}{3}\cos\theta\Big|_{\theta=0}^{\theta=2\pi}$$

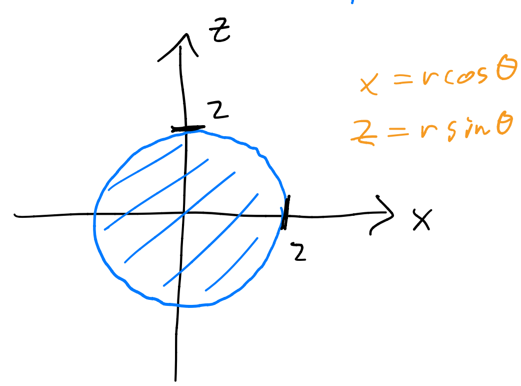
$$=20\pi+\frac{8}{3}\left(\cos(2\pi)-\cos(0)\right)=\boxed{20\pi}$$



side view:



projection onto xz-plane:



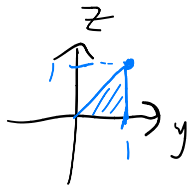
Exercise 4

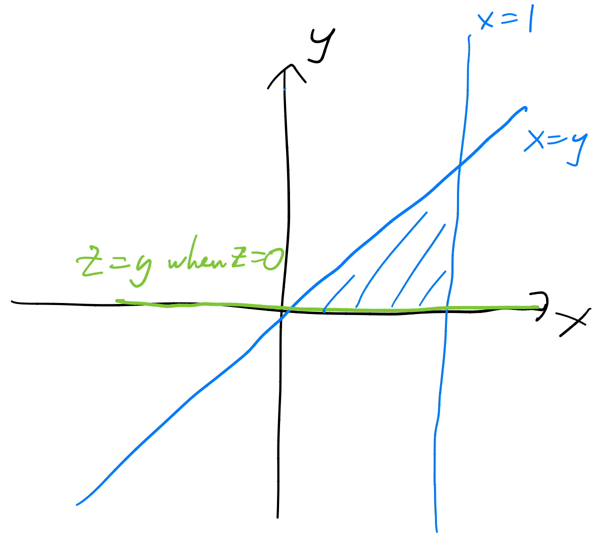
Write the following integral in a few different orders of integration: $\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$

$x=1 \quad z=y$

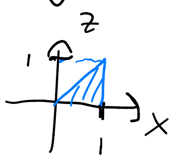
$x=y \quad z=0$

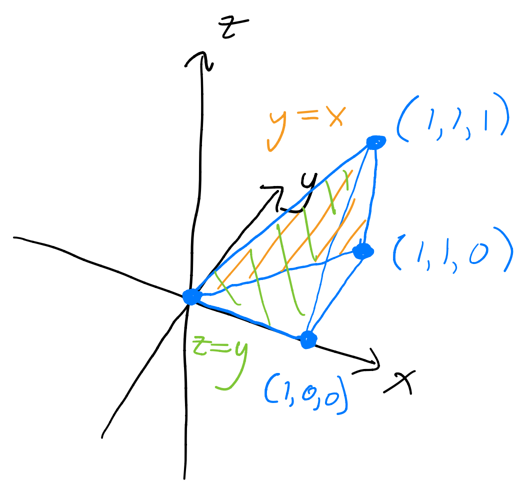
$$\int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz$$





$$\int_0^1 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx$$





We are integrating over this tetrahedron

15.2 — TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

Review

(a) Cylindrical coordinates are "polar coordinates plus a z direction."

(b) Conversion from rectangular to/from cylindrical coordinates:

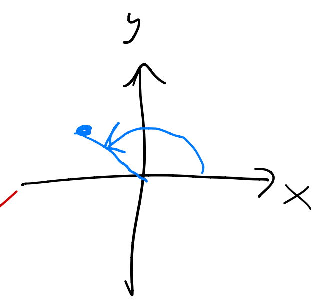
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

Exercise 5

Convert the following points from rectangular to cylindrical coordinates.

(a) $(x, y, z) = (-\sqrt{2}, \sqrt{2}, 1)$.

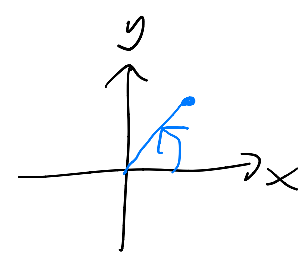


$$r = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2$$

$$\theta = \arctan\left(\frac{\sqrt{2}}{-\sqrt{2}}\right) = \arctan(-1) = \frac{3\pi}{4} \text{ or } \cancel{\frac{7\pi}{4}}$$

$$\boxed{(r, \theta, z) = \left(2, \frac{3\pi}{4}, 1\right)}$$

(b) $(x, y, z) = (2, 2, 2)$.



$$r = \sqrt{2^2 + 2^2} = \sqrt{8}$$

$$\theta = \arctan\left(\frac{2}{2}\right) = \arctan(1) = \pi/4 \text{ or } \cancel{\frac{5\pi}{4}}$$

$$\boxed{(r, \theta, z) = (\sqrt{8}, \pi/4, 2)}$$

Exercise 6

Convert $(r, \theta, z) = (2, \pi/6, -1)$ to rectangular coordinates.

$$x = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$$

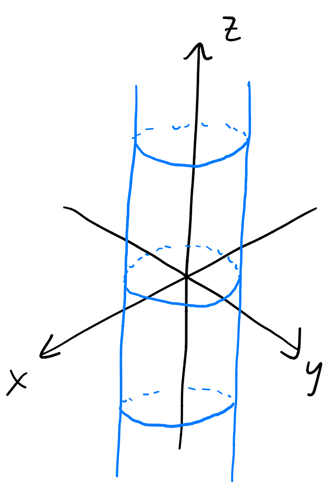
$$y = 2\sin\left(\frac{\pi}{6}\right) = 1$$

$$z = -1$$

$$\boxed{(x, y, z) = (\sqrt{3}, 1, -1)}$$

Exercise 7

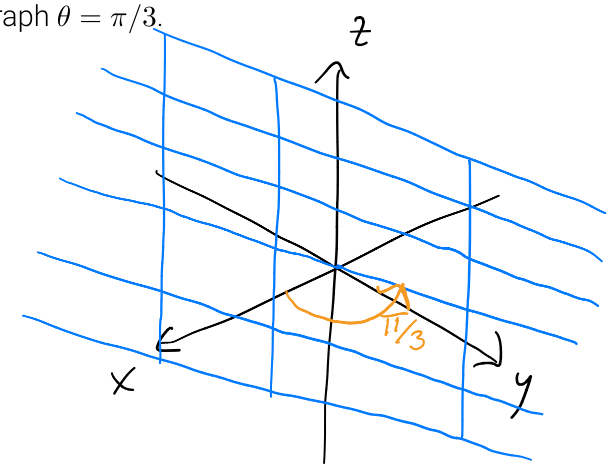
Graph $r = 2$.



cylinder with radius 2

Exercise 8

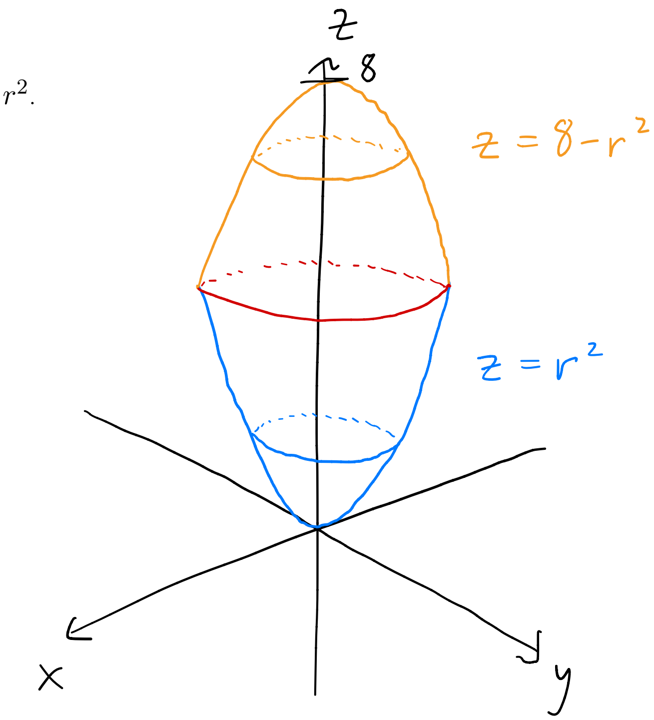
Graph $\theta = \pi/3$.



plane that is vertical and goes in the direction $\theta = \frac{\pi}{3}$.

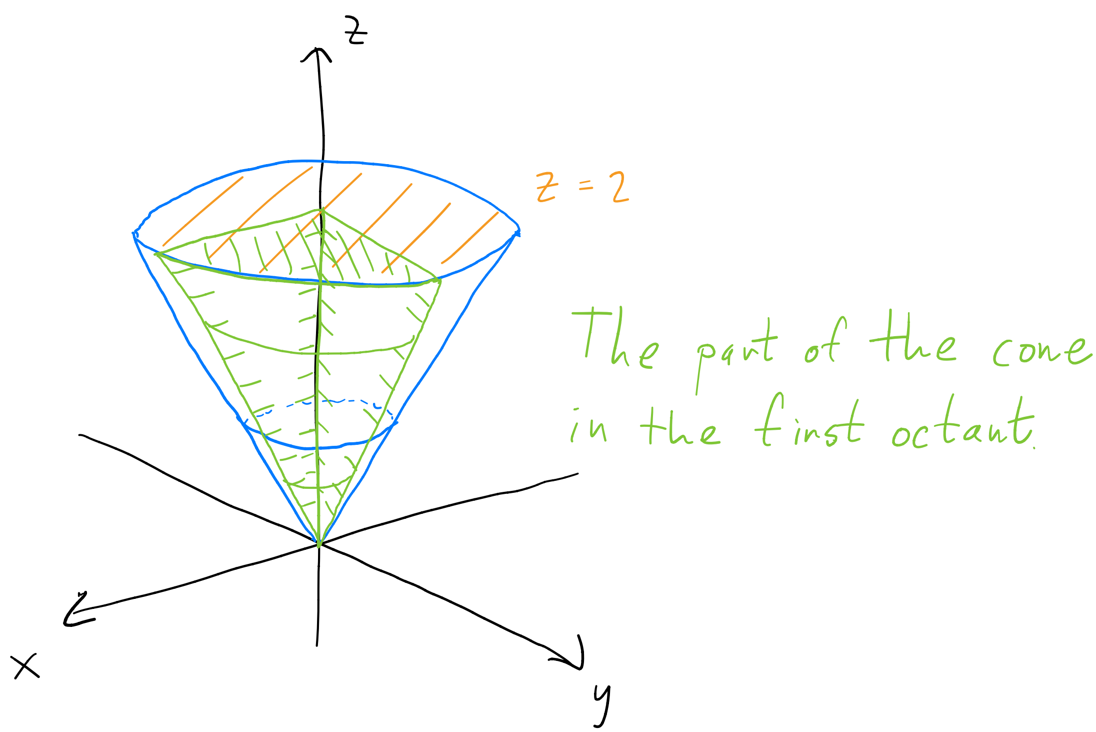
Exercise 9

Sketch the solid $r^2 \leq z \leq 8 - r^2$.



Exercise 10

Sketch the solid $0 \leq \theta \leq \pi/2, \cancel{r \leq z \leq 2.}$ $0 \leq r \leq z \leq 2.$

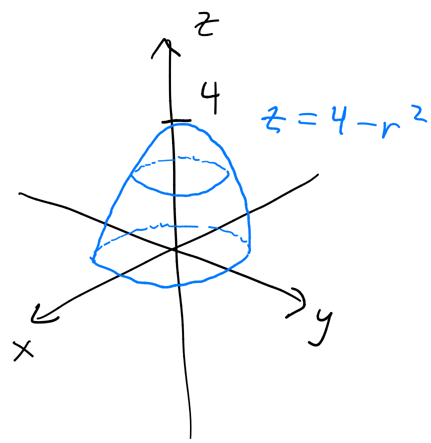


Exercise 11

For the following, set up the integral in cylindrical coordinates, but do not evaluate.

(a) $\iiint_E (x+y+z)\,\mathrm{d}V$, where E is the solid in the first octant under the paraboloid $z = 4 - x^2 - y^2$.

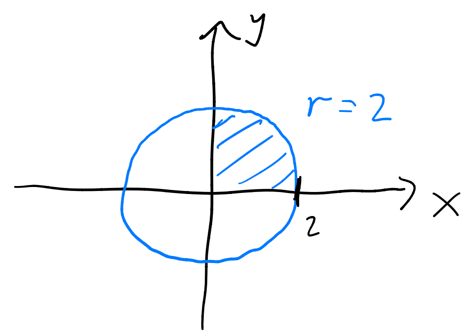
$$\int_0^{\pi/2}\int_0^{2}\int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\, r\, dz\, dr\, d\theta$$



At $z=0$:

$4 - x^2 - y^2 = 0$

$x^2 + y^2 = 4$



Exercise 11 continued on next page...

(b) $\iiint_E (x-y)\,\mathrm{d}V$, where E is the solid that lies between the cylinders $x^2+y^2=1$ and $x^2+y^2=16$, above the xy-plane, and below the plane $z=y+4$.

$$\int_0^{2\pi}\int_1^4\int_0^{r\sin\theta+4}(r\cos\theta - r\sin\theta)\,r\,dz\,dr\,d\theta$$