# EXAM 3 REVIEW

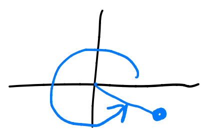
## Exercise 1

Convert $(x, y, z) = (\sqrt{3}, -1, 2\sqrt{3})$ from rectangular to cylindrical and spherical coordinates.

cylindrical:

$$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$$

$$\theta = \arctan\left(\frac{-1}{\sqrt{3}}\right) = \cancel{\frac{5\pi}{6}} \text{ or } \frac{11\pi}{6}$$



$$(r, \theta, z) = \left(2, \frac{11\pi}{6}, 2\sqrt{3}\right).$$

spherical:

$$\rho = \sqrt{(\sqrt{3})^2 + (-1)^2 + (2\sqrt{3})^2} = 4$$

$$\phi = \arccos\left(\frac{2\sqrt{3}}{4}\right) = \frac{\pi}{6}$$

$\theta$ = same as cylindrical

$$(\rho, \theta, \phi) = \left(4, \frac{11\pi}{6}, \frac{\pi}{6}\right).$$

## Exercise 2

Convert $(r, \theta, z) = (1, 2\pi/3, -3)$ to rectangular coordinates.

$$x = r\cos\theta = \cos(2\pi/3) = -\frac{1}{2}$$

$$y = r\sin\theta = \sin(2\pi/3) = \frac{\sqrt{3}}{2}$$

$$(x, y, z) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, -3\right)$$

## Exercise 3

Convert $(\rho, \theta, \phi) = (6, \pi/3, \pi/6)$ to rectangular coordinates.

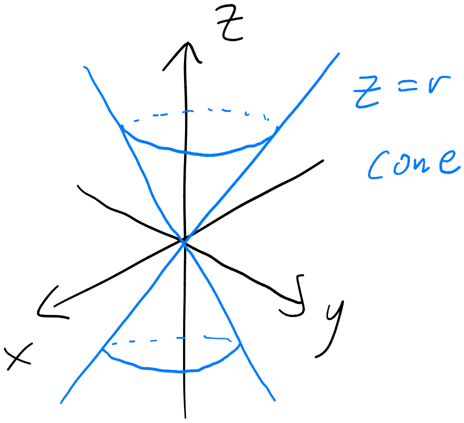
$$x = \rho\sin\phi\cos\theta = 6\sin(\pi/6)\cos(\pi/3) = 6\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

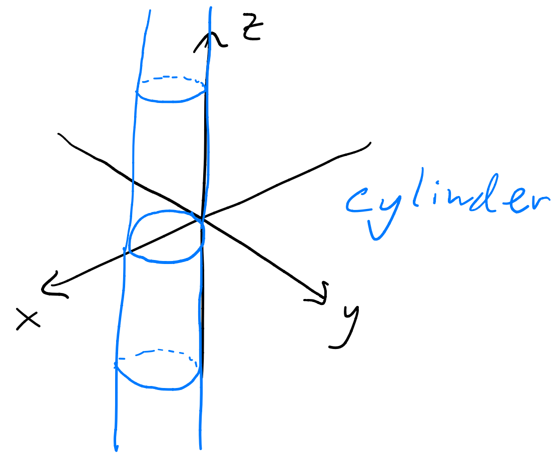
$$y = \rho\sin\phi\sin\theta = 6\sin(\pi/6)\sin(\pi/3) = 6\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$z = \rho\cos\phi = 6\cos(\pi/6) = 6\left(\frac{\sqrt{3}}{2}\right)$$

## Exercise 4

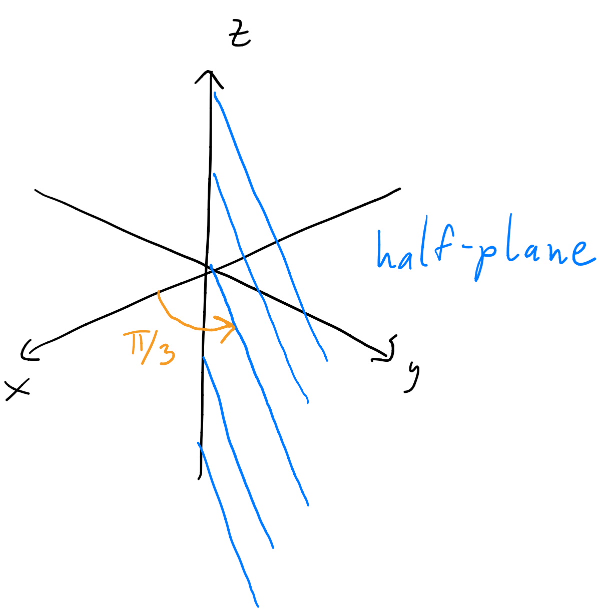
In cylindrical coordinates, graph $z = r$. Also graph $r = \cos(\theta)$.

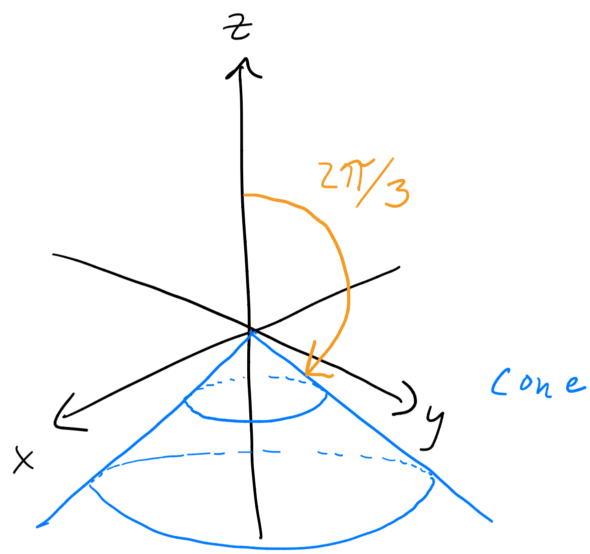



## Exercise 5

In spherical coordinates, graph $\theta = \pi/3$. Also graph $\phi = 2\pi/3$.

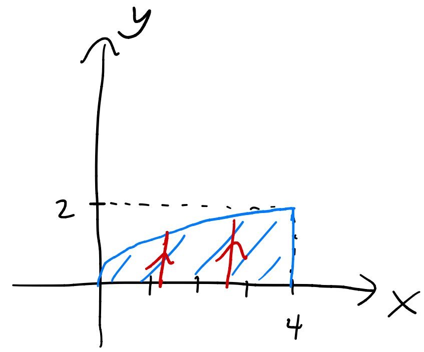


## Exercise 6

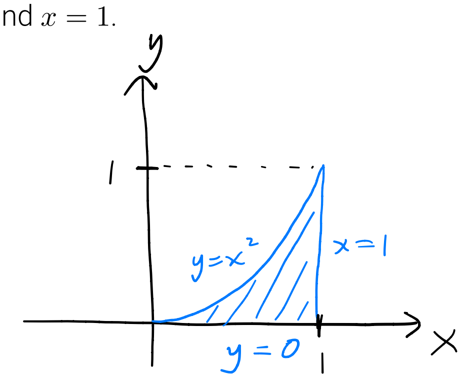
Set up but do not evaluate the following integrals. Set them up in the coordinate system you would try to actually evaluate them in.

(a) $\iint_D \frac{y}{x^2+1}\,\mathrm{d}A$, where $D = \{(x,y) : 0 \le x \le 4, 0 \le y \le \sqrt{x}\}$.




$$\int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2+1}\, dy\, dx$$

(b) $\iint_D x\cos(y)\,\mathrm{d}A$, where $D$ is the region bounded by $y = 0$, $y = x^2$, and $x = 1$.

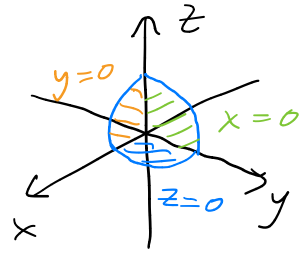



$$\int_0^1 \int_0^{x^2} x\cos(y)\, dy\, dx$$

*Exercise 6 continued on next page. . .*

(c) $\iiint_E xe^{x^2+y^2+z^2}\,\mathrm{d}V$, where $E$ is the portion of the unit ball in the first octant.

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} \rho\sin\phi\cos\theta\, e^{\rho^2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$




(d) Find the area of the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.

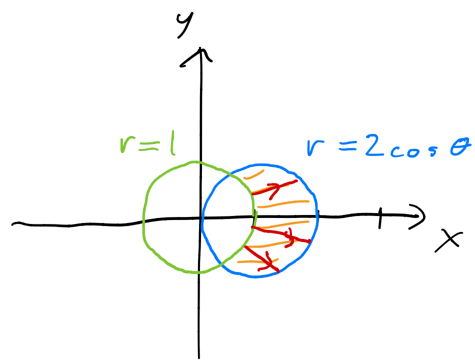
$$\int_{-\pi/3}^{\pi/3}\int_1^{2\cos\theta} 1\cdot r\, dr\, d\theta$$

$x^2-2x+1+y^2=1$

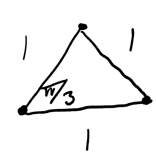
$x^2+y^2=2x$

$r^2=2r\cos\theta$

$r=2\cos\theta$




At what angle do the two circles intersect? $\pi/3$ and $-\pi/3$

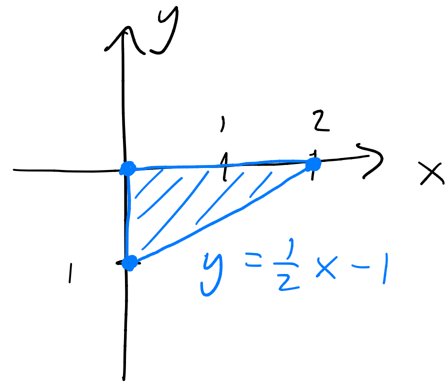



*Exercise 6 continued on next page...*

(e) Find the volume under $z = x^2 + 3y$ and above the triangle in the $xy$-plane with vertices $(0, 0)$, $(2, 0)$, and $\cancel{(-1, 0)}$.

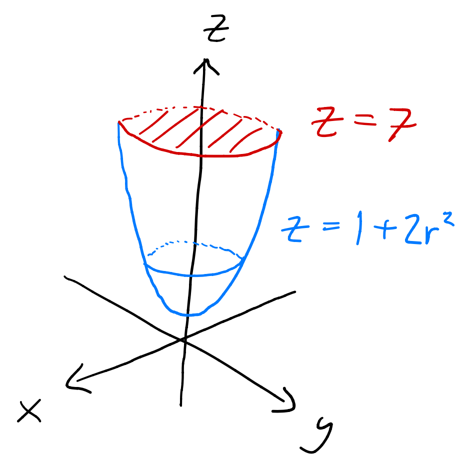
$(0,-1)$

$$\int_0^2 \int_{\frac{1}{2}x-1}^{0} (x^2+3y)\,dy\,dx$$




(f) Find the volume of the region bounded between the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant.

$$\int_0^{\pi/2} \int_0^{\sqrt{3}} \left(7 - (1+2r^2)\right) r\,dr\,d\theta$$



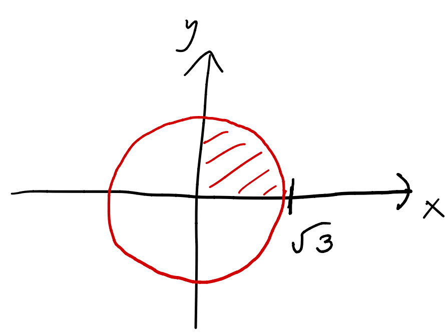

intersection:

$7 = 1 + 2r^2$

$6 = 2r^2$

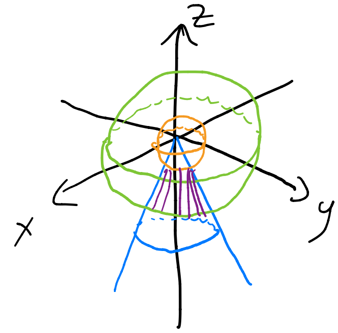
$3 = r^2$

$r = \sqrt{3}$




*Exercise 6 continued on next page...* 

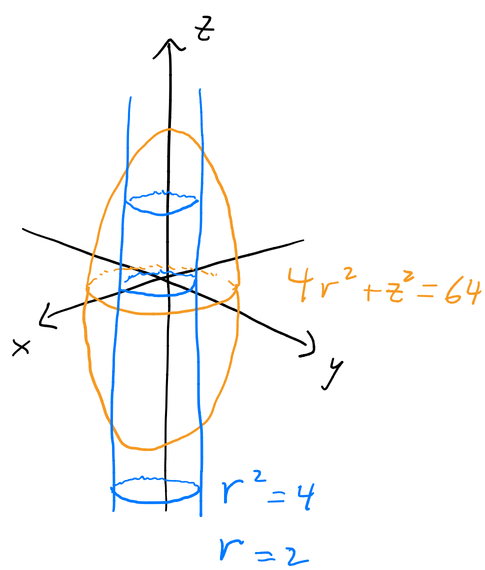
(g) $\iiint_E y^2x^2\,\mathrm{d}V$, where $E$ lies below the cone $\phi = 5\pi/6$ and between the spheres $\rho = 1$ and $\rho = 3$.




$$\int_0^{2\pi}\int_{\frac{5\pi}{6}}^{\pi}\int_1^3 \rho^2\sin^2\phi\sin^2\theta\,\rho^2\sin^2\phi\cos^2\theta\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

(h) Find the volume of the region that is inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$.

$$\int_0^{2\pi}\int_0^{2}\int_{-\sqrt{64-4r^2}}^{\sqrt{64-4r^2}} 1\, r\,dz\,dr\,d\theta$$

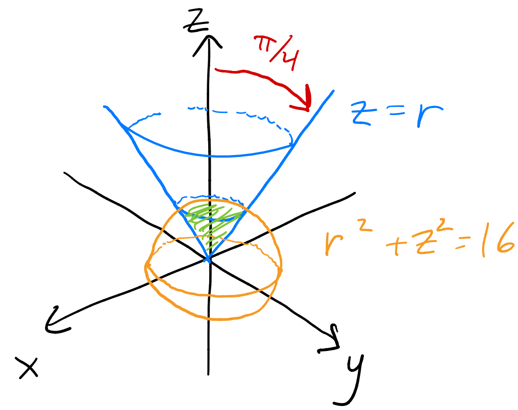



top/bottom of ellipsoid:

$$z^2 = 64-4r^2$$
$$z = \pm\sqrt{64-4r^2}$$

(i) $\iiint_E (x^2+y^2+z^2)^{3/2}\,\mathrm{d}V$, where $E$ lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=16$.

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{4}(\rho^2)^{3/2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



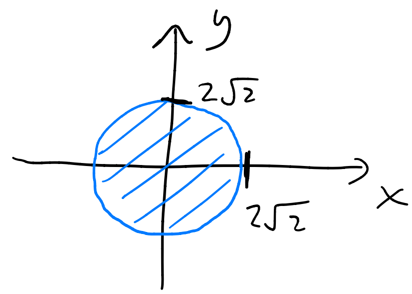

Intersection!

$r^2+r^2=16$

$2r^2=16$

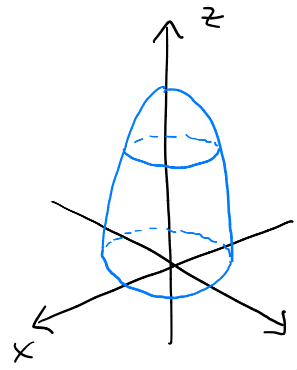
$r^2=8$

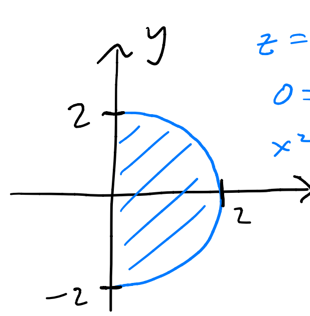
$r=2\sqrt{2}$




(j) $\iiint_E (x+y+z)\,\mathrm{d}V$, where $E$ is the solid under the paraboloid $z=4-x^2-y^2$, above the $xy$-plane, and on the positive $x$ side of the plane $x=0$.

$$\int_{-\pi/2}^{\pi/2}\int_0^{2}\int_0^{4-r^2}(r\cos\theta+r\sin\theta+z)\,r\,dz\,dr\,d\theta$$




$z=0$ cross section:

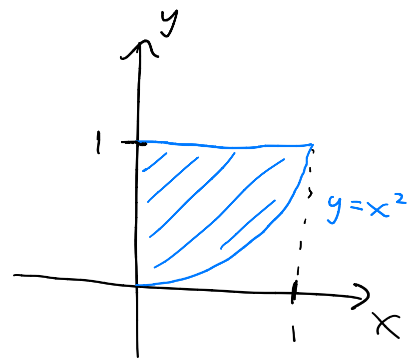
$0=4-x^2-y^2$

$x^2+y^2=4$

## Exercise 7

Evaluate the following.

(a) $\int_0^1 \int_{x^2}^1 \sqrt{y}\sin(y)\,\mathrm{d}y\,\mathrm{d}x$




$$= \int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\sin(y)\,dx\,dy$$

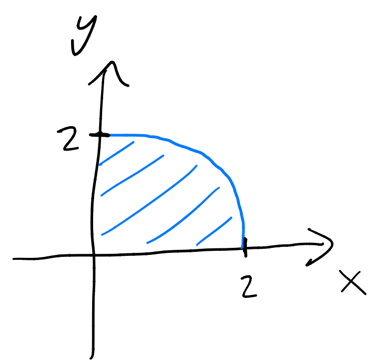
$$= \int_0^1 y\sin(y)\,dy \qquad u = y \quad dv = \sin(y)\,dy \qquad du = dy \quad v = -\cos(y)$$

$$= -y\cos(y)\Big|_0^1 - \int_0^1 (-\cos(y))\,dy$$

$$= -\cos(1) + \sin(y)\Big|_0^1$$

$$= -\cos(1) + \sin(1).$$

(b) $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,\mathrm{d}y\,\mathrm{d}x$




$$= \int_0^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta \qquad u = -r^2 \quad du = -2r\,dr$$

$$= \int_0^{\pi/2} -\frac{1}{2}\int_0^{-4} e^u\,du\,d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{2} e^u\Big|_{u=0}^{u=4}\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} (e^4 - 1)\,d\theta$$

$$= \boxed{-\frac{\pi}{4}(e^4 - 1).}$$

## Exercise 8

Find the mass of a ball of radius 3 if its density is proportional to the distance from the center of the ball. (Assume that the constant of proportionality is 1.) [Note: not all instructors covered this topic.]

$$\text{density}(\rho, \theta, \phi) = \rho.$$

$$\text{mass} = \iiint_B \text{density}\, dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{3} \rho \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_0^{\pi} \sin\phi \, d\phi \int_0^3 \rho^3 \, d\rho$$

$$= 2\pi (-\cos\phi)\Big|_{\phi=0}^{\phi=\pi} \frac{1}{4}\rho^4\Big|_{\rho=0}^{\rho=3}$$

$$= -2\pi(-1-1)\frac{1}{4} 3^4$$

$$= \boxed{3^4 \pi}$$