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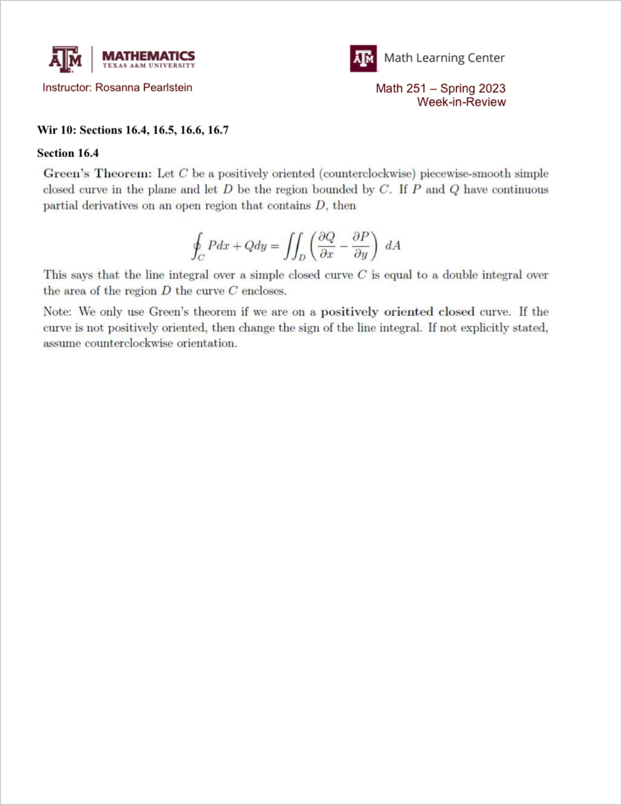
Monday, April 24, 2023 5:15 PM



MATHEMATICS
TEXAS A&M UNIVERSITY

Instructor: Rosanna Pearlstein

Math Learning Center

Math 251 – Spring 2023
Week-in-Review

Wir 10: Sections 16.4, 16.5, 16.6, 16.7

Section 16.4

Green's Theorem: Let C be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

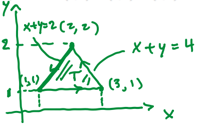
$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

This says that the line integral over a simple closed curve C is equal to a double integral over the area of the region D the curve C encloses.

Note: We only use Green's theorem if we are on a **positively oriented closed** curve. If the curve is not positively oriented, then change the sign of the line integral. If not explicitly stated, assume counterclockwise orientation.

$P=y^2$ $Q=x$

Problem 1 Evaluate $\oint_C y^2\,dx + x\,dy$ where C is the triangular path from $(1,1)$ to $(3,1)$ to $(2,2)$ then back to $(1,1)$.



We apply Green's Theorem:

$$\iint_T Q_x - P_y \, dA =$$

$$= \int_1^2 \int_{2-x}^{4-x} (1-2y)\,dy\,dx$$

$$y - y^2 \Big|_{y=2-x}^{4-x} \qquad \int (4-x) - (4-x)^2 - (2-x) + (2-x)^2 \, dx$$

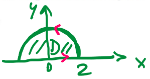
$$-\frac{1}{2}(4-x)^2 + \frac{1}{3}(4-x)^3 + \frac{1}{2}(2-x)^2 - \frac{1}{3}(2-x)^3 \Big|_{x=1}^{2}$$

$$-\frac{1}{2}(4) + \frac{1}{3}(8) + 0 - 0 - \left(-\frac{1}{2}(9) + \frac{1}{3}(27) + \frac{1}{2}(1) - \frac{1}{3}(1)\right) =$$

$$= -2 + \frac{8}{3} + \frac{9}{2} + 9 + \frac{1}{2} - \frac{1}{3}$$

$P=y^2$ $Q=x^2$

Problem 2 Evaluate $\oint_C y^2\,dx + x^2\,dy$ where C is the boundary of the region bounded by the semicircle $y=\sqrt{4-x^2}$ and the x axis. Assume positive orientation.



Apply Green's Thm:

$$\iint (Q_x - P_y)\,dA =$$

$Q_x = 2x$

$P_y = 2y$

polar coordinates

D

$$= \int_0^{\pi}\int_0^{2} 2(\cos\theta - \sin\theta)\, r\, dr\, d\theta =$$

$$= \left[r^2\right]_0^2 \left[\sin\theta + \cos\theta\right]_0^{\pi} =$$

$$= (4-0)\left(0-1-(0+1)\right) =$$

$$= 4(-2) = -8.$$

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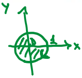
Problem $\boxed{3}$ Suppose a particle travels one revolution clockwise around the unit circle under the force field $\mathbf{F}(x, y) = \langle \underbrace{e^x - y^3}_{P}, \underbrace{\cos(y) + x^3}_{Q} \rangle$. Find the work done.

GREEN'S THM



$$Q_x - P_y = 3x^2 - (-3y^2)$$

$$\iint_{D_1} 3(x^2+y^2)\, dA = \int_0^{2\pi}\int_0^{1} 3\underbrace{r^2}\, r\, dr\, d\theta =$$

$$= 2\pi\ 3\ \frac{r^4}{4}\Big|_0^1 = \frac{3\pi}{2}$$

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Section 16.5

Definition: The **del operator**, denoted by $\vec{\nabla}$, is defined as $\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

Definition of curl and divergence:

$\text{curl}(\vec{F}) = \vec{\nabla} \times \vec{F}$

$\text{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F}$

Problem $\boxed{4}$ Find the divergence and curl of $\mathbf{F} = \langle xy, xz, xyz^2 \rangle$.

$$\text{div}\,\vec{F} = \frac{\partial xy}{\partial x} + \frac{\partial xz}{\partial y} + \frac{\partial xyz^2}{\partial z} = y + 0 + 2xyz$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy & xz & xyz^2 \end{vmatrix} = (xz^2 - x)\hat{i} - (yz^2 - 0)\hat{j} + (x - z)\hat{k}$$

Theorem: If $\mathbf{F}$ is a vector field defined on all of $\Re^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field. This gives us a way to determine whether a vector function on $\Re^3$ is conservative.

Problem $\boxed{5}$ If $\mathbf{F} = \langle x,\ e^y \sin z,\ e^y \cos z \rangle$, Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{r}(t) = \langle t^4, t, 2t^2 \rangle$, for $1 \le t \le 2$.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & e^y \sin z & e^y \cos z \end{vmatrix} = 0\hat{i} - 0\hat{j} + 0\hat{k} \Rightarrow \vec{F} \text{ is conserv.}$$

Find f such that $\vec{F} = \vec{\nabla} f = \langle f_x, f_y, f_z \rangle$

$f_x = x \Rightarrow f = \frac{x^2}{2} + g(y,z)$

$f_y = e^y \sin z \Rightarrow f = e^y \sin z + h(x,z)$

$f_z = e^y \cos z \Rightarrow f = e^y \sin z + g(x,y)$

$\Big\} f(x,y,z) = \frac{x^2}{2} + e^y \sin z + C$

Check!

By FTLI, $\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(1)) =$

$= \frac{2^8}{2} + e^2 \sin 8 - \frac{1}{2} - e \sin 2$

$\vec{r}(2) = (2^4, 2, 2 \cdot 4)$

$\vec{r}(1) = (1, 1, 2)$

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Problem $\boxed{6}$ If time permits, discuss what operaters make sense (similar to webassign section 16.5 problem 8)

Which of the following do/do not make sense, if f is a differentiable function $= f(x,y,z)$ and $\vec{F}$ is a differentiable vector field, $\vec{F}(x,y,z)$?

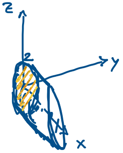
curl $\vec{F}$ yes
grad $\vec{F}$ no
div $\vec{F}$ yes
curl (grad f) yes
div (curl $\vec{F}$) yes
curl $(\vec{\nabla} f)$ yes
div $(\vec{\nabla} f)$ yes

curl f no
grad f yes
div f no
grad (curl $\vec{F}$) no
curl (div $\vec{F}$) no
grad (curl $\vec{F}$) no
grad (div $\vec{F}$) yes

Pb. 7

Evaluate $\iint_S 2y\,dS$ where S is the portion of $y^2+z^2=4$ between $x=0$ and $x=3-z$.

NO TOP NOR BOTTOM



$\vec{r}(\theta, x) = \langle x, 2\cos\theta, 2\sin\theta \rangle$

$$\begin{array}{c|ccc|} & \hat{i} & \hat{j} & \hat{k} \\ r_\theta & 0 & -2\sin\theta & 2\cos\theta \\ r_x & 1 & 0 & 0 \end{array} \Rightarrow r_\theta \times r_x = 0\hat{i} + 2\cos\theta\,\hat{j} + 2\sin\theta\,\hat{k}$$

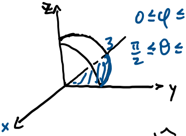
$|r_\theta \times r_x| = 2.$

$$\int_0^{2\pi}\int_0^{2} 4\cos\theta \cdot 2\,dx\,d\theta = \int_0^{2\pi} 16\cos\theta\,d\theta = 16\sin\theta\Big|_0^{2\pi} = 0$$

Pb. 8

Evaluate $\iint_S xz\,dS$ where S is the portion of the sphere of radius 3 with $x \le 0$, $y \ge 0$ and $z \ge 0$.

$$\vec{r}(\theta,\varphi)=3\sin\varphi\cos\theta\,\hat{\imath}+3\sin\varphi\sin\theta\,\hat{\jmath}+3\cos\varphi\,\hat{k}$$



$0 \le \varphi \le \frac{\pi}{2}$

$\frac{\pi}{2} \le \theta \le \pi$

$$\vec{r}_\theta\times\vec{r}_\varphi=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -3\sin\varphi\sin\theta & 3\sin\varphi\cos\theta & 0\\ 3\cos\varphi\cos\theta & 3\cos\varphi\sin\theta & -3\sin\varphi\end{vmatrix}$$

$$\underline{-9\sin^2\varphi\cos\theta}\,\hat{\imath}\ \underline{-9\sin^2\varphi\sin\theta}\,\hat{\jmath}+\left(-9\sin\varphi\cos\varphi\sin^2\theta-9\sin\varphi\cos\varphi\cos^2\theta\right)\hat{k}$$

$$\underline{-9\sin\varphi\cos\varphi}\,\hat{k}$$

$$|\vec{r}_\theta\times\vec{r}_\varphi|=9\sin\varphi$$

$$\int_{\frac{\pi}{2}}^{\pi}\int_0^{\pi/2}3\sin\varphi\cos\theta\cdot3\cos\varphi\;d\varphi\,d\theta=9\int_{\frac{\pi}{2}}^{\pi}\cos\theta\,d\theta*\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi\,d\varphi=$$

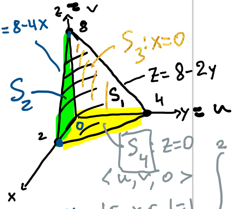
$$=9\left[\sin\theta\right]_{\frac{\pi}{2}}^{\pi}*\left[\frac{1}{2}\sin^2\varphi\right]_0^{\frac{\pi}{2}}=$$

$$=9(0-1)\left(\frac{1}{2}1-0\right)=-\frac{9}{2}$$

Pb. 9 SET UP ONLY. *Note that all four surfaces are included.*

S_1

Evaluate $\iint_S yz + 4xy\,dS$ where S is the surface of the solid bounded by $4x + 2y + z = 8$, $z = 0$, $y = 0$ and $x = 0$.



S_4: $z=0$

$\langle u, v, 0\rangle$

$|r_u \times r_v| = 1$

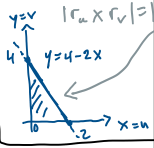
$$\int_0^2\int_0^{4-2u} 4uv\,dr\,du$$

$\boxed{S_1}$ $\vec{r}(u,v) = \langle u, v, 8-4u-2v\rangle$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -4 \\ 0 & 1 & -2 \end{vmatrix} = \langle 4, +2, 1\rangle$$

$$|r_u \times r_v| = \sqrt{16+4+1} = \sqrt{21}$$

$$\int_0^2\int_0^{4-2u} \left(8v - 4\cancel{u}v - 2v^2 + 4\cancel{u}v\right) dv\,du \quad \boxed{S_1}$$



$\boxed{S_3}$ $r(u,v) = \langle 0, u, v\rangle$

$r_u \times r_v = \langle \pm 1, 0, 0\rangle$, $|r_u \times r_v| = 1$

$$\int_0^4\int_0^{8-2u} uv - 0\,dv\,du$$

$\boxed{S_2}$ $\vec{r}(u,v) = \langle u, 0, v\rangle$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0\hat{i} - 1\hat{j} + 0\hat{k}$$

$|\vec{r}_u \times r_v| = 1$

$$\int_0^2\int_0^{8-4x} 0\,dA = 0$$

Answer to pb. 9:

$$\iint_S yz + 4xy\,dS = \left(\frac{64\sqrt{21}}{3}\right) + (0) + \left(\frac{128}{3}\right) + \left(\frac{32}{3}\right) = \frac{64\sqrt{21}}{3} + \frac{160}{3} = \boxed{151.0949}$$

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Pb. 10

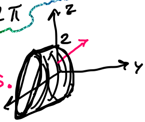
Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = x^2\,\vec{i} + 2z\,\vec{j} - 3y\,\vec{k}$ and S is the portion of $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$ oriented outwards (i.e. away from the x-axis).

$\vec{r}(x,\theta) = \langle x, 2\cos\theta, 2\sin\theta\rangle$ $\left\{\begin{array}{l} 0 \le x \le 3 - 2\sin\theta \\ 0 \le \theta \le 2\pi \end{array}\right.$

$\vec{r}_x \times \vec{r}_\theta = \langle 0, 2\cos\theta, 2\sin\theta\rangle$

Is this the "+" orientation? Yes.

$\vec{F}(\vec{r}(x,\theta)) = x^2\hat{i} + 4\sin\theta\hat{j} - 6\cos\theta\hat{k}$



$$\vec F\cdot(\vec r_x \times \vec r_\theta) = 0 + 8\sin\theta\cos\theta - 12\sin\theta\cos\theta$$

$$\int_0^{2\pi}\int_0^{3-2\sin\theta} -4\sin\theta\cos\theta \, dx \, d\theta$$

$$-4\sin\theta\cos\theta\,(3-2\sin\theta)$$

$$\int_0^{2\pi} -12\sin\theta\cos\theta + 8\sin^2\theta\cos\theta \, d\theta$$

$$\left[-12\,\frac{1}{2}\sin^2\theta + 8\,\frac{1}{3}\sin^3\theta\right]_0^{2\pi} = 0$$

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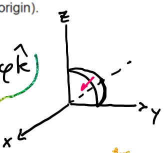
Pb. 11

Evaluate $\iint_S \vec F \cdot d\vec S$ where $\vec F = \vec i + z\vec j + 6x\vec k$ and S is the portion of the sphere of radius 3 with $x \le 0$, $y \ge 0$ and $z \ge 0$ oriented inward (i.e. towards the origin).



$$\vec r(\theta,\varphi) = \langle 3\sin\varphi\cos\theta,\ 3\sin\varphi\sin\theta,\ 3\cos\varphi \rangle$$

$$\vec r_\theta \times \vec r_\varphi = -9\sin^2\varphi\cos\theta\,\hat\imath - 9\sin^2\varphi\sin\theta\,\hat\jmath - 9\sin\varphi\cos\varphi\,\hat k$$

Is this the "+" orientation? NOPE

(so we will dot product with $-r_\theta \times r_\varphi$)

$$\vec F(\vec r(\theta,\varphi)) = \hat\imath + 3\cos\varphi\,\hat\jmath + 18\sin\varphi\cos\theta$$

dot product

$$F\cdot(\vec r_\theta \times \vec r_\varphi) = +9\sin^2\varphi\cos\theta + 27\sin^2\varphi\cos\varphi\sin\theta + 162\sin^2\varphi\cos\varphi\cos\theta$$

$$\int_{\frac{\pi}{2}}^{\pi}\int_0^{\pi/2} 9\sin^2\varphi\cos\theta + \underbrace{\sin^2\varphi\cos\varphi}(27\sin\theta + 162\cos\theta)\, d\theta$$

$$\int \frac{9}{2}(1-\cos 2\varphi)\int\cos\theta + \frac{\sin^3\varphi}{3}\Big|_0^{\frac{\pi}{2}} \left(-27\cos\theta + 162\sin\theta\right)_{\frac{\pi}{2}}^{\pi}$$

$$\frac{9}{2}\left(\varphi - \frac{1}{2}\sin 2\varphi\right)_{\varphi=0}^{\pi/2} (\sin\theta)_{\frac{\pi}{2}}^{\pi} +$$

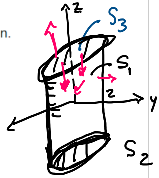
$$\frac{9}{2}\left(\frac{\pi}{2}-0\right)(0-1)+\left(\frac{1}{3}-0\right)(27+0-0-162)$$

$$\frac{9\pi}{4}+\frac{1}{3}(27-162)$$

Pb. 12 SET UP ONLY. *Note that all three surfaces are included.*

Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}=\underset{\smile}{yz}\,\vec{i}+x\,\vec{j}+3y^2\,\vec{k}$ and S is the surface of the solid bounded by $x^2+y^2=4$, $\overset{\text{lower}}{z=x-3}$, and $\overset{\text{upper}}{z=x+2}$ with the negative orientation.

Note that all three surfaces of this solid are included in S.



$\boxed{S_1}$ $\vec{r}(\theta,z)=\langle 2\cos\theta, 2\sin\theta, z\rangle$

$$\vec{r}_\theta\times\vec{r}_z=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2\sin\theta & 2\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=\langle 2\cos\theta, 2\sin\theta, 0\rangle$$

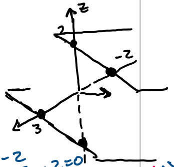
This is the "+" orientation

dot

$$\vec{F}(\vec{r}(\theta,z))=2(\sin\theta)z\,\hat{i}+2\cos\theta\,\hat{j}+12\sin^2\theta\,\hat{k}$$

$$\vec{F}\cdot(r_\theta\times r_z)=4z\sin\theta\cos\theta+4\cos\theta\sin\theta+0$$

$$-\int_0^{2\pi}\int_{2\cos\theta-3}^{2\cos\theta+2}4\sin\theta\cos\theta(z+1)\,dz\,d\theta$$



$x-z-3=0$

$\boxed{S_2}$: $z=x-3$ $\quad\langle u, v, u-3\rangle$

$$\vec{r}_u\times\vec{r}_v=\langle 1, 0, -1\rangle$$

POSITIVE (OUTWARD) orientation

dot

$$\vec{F}(S_2)=v(u-3)\,\hat{i}+u\,\hat{j}+3v^2\,\hat{k}$$

$$\vec{F}\cdot r_u\times r_v=v(u-3)-3v^2 \quad \text{POLAR}$$

$u=r\cos\theta \qquad v=r\sin\theta$

$$\boxed{S_2}\ -\int_0^{2\pi}\int_0^2\left(r\sin\theta(r\cos\theta-3)-3r^2\sin^2\theta\right)r\,dr\,d\theta$$

S_3: $z=x-2$

$x-z-2=0$

$\vec{r}=\langle u, v, u-2\rangle \qquad r_u\times r_v=\langle -1, 0, +1\rangle$

$$\vec{F}(S_3)=v(u-2)\,\hat{i}+u\,\hat{j}+3v^2\,\hat{k}$$

$$\vec{F}\cdot\vec{r}_u\times\vec{r}_v=-v(u-2)+3v^2$$

$$-\int_0^{2\pi}\int_0^2\left(-r\sin\theta(r\cos\theta-2)-3r^2\sin^2\theta\right)r\,dr\,d\theta.$$