MATHEMATICS
TEXAS A&M UNIVERSITY

Instructor: Rosanna Pearlstein

Math Learning Center

Math 251 – Spring 2023
Week-in-Review

Wir 2: 12.4, 12.5, 12.6

Section 12.4

1. Find the cross product of $\langle 1,1,3\rangle$ and $\langle -2,-1,5\rangle$ and find the area of the parallelogram determined by the two vectors.

$\vec{a} = \langle 1,1,3\rangle$, $\vec{b} = \langle -2,-1,5\rangle$

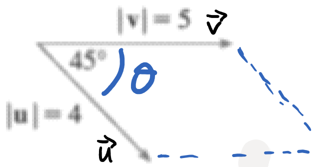
$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 3 \\ -2 & -1 & 5 \end{vmatrix} = \langle 5+3,\ -(5+6),\ -1+2\rangle =$$

$$= \langle 8, -11, 1\rangle$$

$$\begin{array}{r} 122 \\ 64 \\ \hline 186 \end{array}$$

$$\text{Area} = |\vec{a}\times\vec{b}| = \sqrt{64+121+1} = \sqrt{186}$$

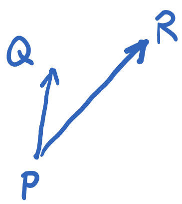
2. Find $|\mathbf{u}\times\mathbf{v}|$ and determine if $\mathbf{u}\times\mathbf{v}$ points in or out of the page.



$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta =$$

$$= 4\cdot 5\sin\frac{\pi}{4} = 20\,\frac{\sqrt{2}}{2} = 10\sqrt{2}.$$

3. Find two unit vectors that are orthogonal to the plane that passes through the points $P(1,0,1)$, $Q(2,3,4)$ and $R(2,1,1)$.



$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 3 \\ 1 & 1 & 0 \end{vmatrix} = \langle -3, -(-3), -2 \rangle =$$

$$= \langle -3, 3, -2 \rangle = \vec{n}$$

$$\frac{\vec{n}}{|\vec{n}|} = \frac{\langle -3, 3, -2 \rangle}{\sqrt{9+9+4}} = \left\langle -\frac{3}{\sqrt{22}}, \frac{3}{\sqrt{22}}, -\frac{2}{\sqrt{22}} \right\rangle.$$

$$-\frac{\vec{n}}{|\vec{n}|} = \left\langle \frac{3}{\sqrt{22}}, \frac{-3}{\sqrt{22}}, \frac{2}{\sqrt{22}} \right\rangle$$

4. Determine whether each expression is meaningful or meaningless (circle one). If so, state whether the expression is a vector or a scalar.

a.) $\mathbf{a} \cdot \mathbf{b}$	meaningful (vector or scalar)	meaningless
b.) $\mathbf{a} \times \mathbf{b}$	meaningful (vector or scalar)	meaningless
c.) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$	meaningful (vector or scalar)	meaningless
d.) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$	meaningful (vector or scalar)	meaningless
e.) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$	meaningful (vector or scalar)	meaningless
f.) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$	meaningful (vector or scalar)	meaningless
g.) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$	meaningful (vector or scalar)	meaningless
h.) $\lvert\mathbf{a}\rvert(\mathbf{b} \times \mathbf{c})$	meaningful (vector or scalar)	meaningless

Section 12.5

1. Find vector, parametric, and symmetric equations for the line through the point $\overset{P_0}{(1, 0, -3)}$ and parallel to the vector $2\,\boldsymbol{i} - 4\,\boldsymbol{j} + 5\,\boldsymbol{k}$. $= \vec{v}$

$P(t) = P_0 + t\,\vec{v}$

$\langle x, y, z \rangle = \langle 1, 0, -3 \rangle + \langle 2t, -4t, 5t \rangle$ vector equ.

param. equs. $\begin{cases} x = 1 + 2t \\ y = 0 - 4t \\ z = -3 + 5t \end{cases}$ $\rightarrow$ $t = \dfrac{x-1}{2} = \dfrac{-y}{4} = \dfrac{z+3}{5}$ symmetric equs.

2. Find parametric and symmetric equations of the line through the points $\overset{A}{(1, 2, 0)}$ and $\overset{B}{(-5, 4, 2)}$.

$P(t) = A + t\,\vec{AB}$ $\qquad \vec{AB} = B - A = \langle -6, 2, 2 \rangle = \vec{v}$

param. equs. $\begin{cases} x = 1 - 6t \\ y = 2 + 2t \\ z = 0 + 2t \end{cases}$ $\quad t = \dfrac{x-1}{-6} = \dfrac{y-2}{2} = \dfrac{z}{2}$ symm. equs.

3. Find parametric and symmetric equations of the line passing through the point $\overset{P_0}{(-3, 5, 4)}$ and parallel to the line $\underbrace{x = 1 + 3t, y = -1 - 2t, z = 3 + t}$.

$P_0(1,-1,3)$ $\vec{v} = \langle 3, -2, 1 \rangle$

param. eqns.
$$\begin{cases} x = -3 + 3t \\ y = 5 - 2t \\ z = 4 + t \end{cases}$$

$$t = \frac{x+3}{3} = \frac{y-5}{-2} = \frac{z-4}{1}.$$

4. Find an equation of the plane through the point $\overset{P_0}{(-4, 3, 1)}$ that is perpendicular to the vector $\boldsymbol{a} = -4\,\boldsymbol{i} + 7\,\boldsymbol{j} - 2\,\boldsymbol{k} = \vec{n}$

$$-4(x+4) + 7(y-3) - 2(z-1) = 0$$

$$\boxed{-4x + 7y - 2z - 35 = 0}$$

$-16 - 21 + 2$

$-37 + 2$

5. Find an equation of the plane passing through the points (1, 2, -3), (2, 3, 1), and (0, -2, -1)

A B C

$= P_0$

$$\vec{n} = \underbrace{\vec{BA}}_{A-B} \times \underbrace{\vec{BC}}_{C-B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & -4 \\ -2 & -5 & -2 \end{vmatrix} = \langle -18, -(-6), 3 \rangle$$

$$= \langle -18, 6, 3 \rangle = \vec{n}$$

$$\boxed{-18(x-0)+6(y+2)+3(z+1)=0}$$

$18 - 6 - 12$ ✓

$36 - 30 - 6$ ✓

6. Determine whether the planes $3x + y - 4z = 3$ and $-9x - 3y + 12z = 4$ are orthogonal, parallel, or neither. Find the angle of intersection and the set of parametric equations for the line of intersection of the planes.

P_1 P_2

$\vec{n}_1 = \langle 3, 1, -4 \rangle$, $\vec{n}_2 = \langle -9, -3, 12 \rangle$ } $\vec{n}_2 = -3\vec{n}_1$ so the planes are parallel. no intersection.

7. Determine whether the planes $x - 3y + 6z = 4$ and $5x + y - z = 4$ are orthogonal, parallel, or neither. Find the angle of intersection and the set of parametric equations for the line of intersection of the planes. ORTHOGONAL? $\vec{n}_1 \cdot \vec{n}_2 = 5 - 3 - 6 \neq 0$ so not orthogonal.

$\vec{n}_1 = \langle 1, -3, 6 \rangle$, $\vec{n}_2 = \langle 5, 1, -1 \rangle \Rightarrow$ not parallel.

If two planes in $\mathbb{R}^3$ are not parallel, they intersect along a line.
Find that line

Step 1: Find a point which is on both planes. (Eliminate y)

$$\begin{cases} x - 3y + 6z = 4 \\ 5x + y - z = 4 \end{cases} \xrightarrow{(3)} \begin{cases} x - 3y + 6z = 4 \\ 15x + 3y - 3z = 12 \end{cases}$$

$16x + 3z = 16$

If $z = 0 \Rightarrow x = 1 \Rightarrow 1 - 3y + 0 = 4$

$-3y = 3$

$y = -1$

$P_0(1, -1, 0)$

Step 2: Find $\vec{n}$ to plane

$$\vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 6 \\ 5 & 1 & -1 \end{vmatrix} = \langle -3, -(-31), 16 \rangle = \langle -3, 31, 16 \rangle = \vec{n}$$

$\langle -3, 31, 16 \rangle$ is the $\vec{v}$ of the line

Step 3: Give plane equ.

This is the plane through P_0, $\perp$ to both planes.

$$\boxed{-3(x-1) + 31(y+1) + 16(z-0) = 0}$$

STEP 2:

$$\begin{cases} x = 1 - 3t \\ y = -1 + 31t \\ z = 0 + 16t \end{cases}$$

8. Find the point where the line $x = \underbrace{1+t}_{-4}$, $y = \underbrace{2t}_{2}$, and $z = \underbrace{-3t}_{-4}$ intersects the plane with equation $-4x + 2y - 4z + 2 = 0$

$-4 - \cancel{4t} + \cancel{4t} + 12t + 2 = 0 \Rightarrow 12t - 2 = 0 \Rightarrow t = \frac{2}{12} = \frac{1}{6}$

plug in the line eqn: $\left(1+\frac{1}{6},\ 2\left(\frac{1}{6}\right),\ -3\left(\frac{1}{6}\right)\right)$

$\left(\frac{7}{6},\ \frac{1}{3},\ -\frac{1}{2}\right)$

9. Find the distance between point (1, 2, 3) and the plane with equation $2x - y + z - 4 = 0$

$$d = \frac{|\cancel{2}\cdot 1 - \cancel{2} + 3 - 4|}{\sqrt{4+1+1}} = \frac{1}{\sqrt{6}} \text{ OR } \frac{\sqrt{6}}{6}$$

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Section 12.6

1. Identify and sketch the following quadric surfaces:

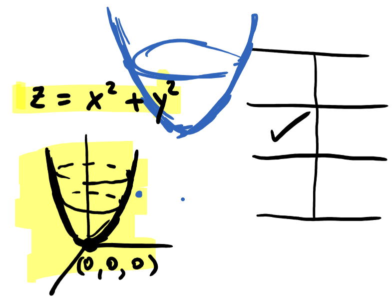
a) $z = (x+4)^2 + (y-2)^2 + 5.$

Let $x = x+4$, $y = y-2$

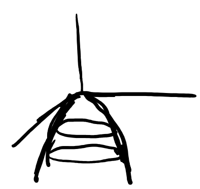
$z = x^2 + y^2 + 5$

Elliptic Paraboloid.

V(−4, 2, 5)

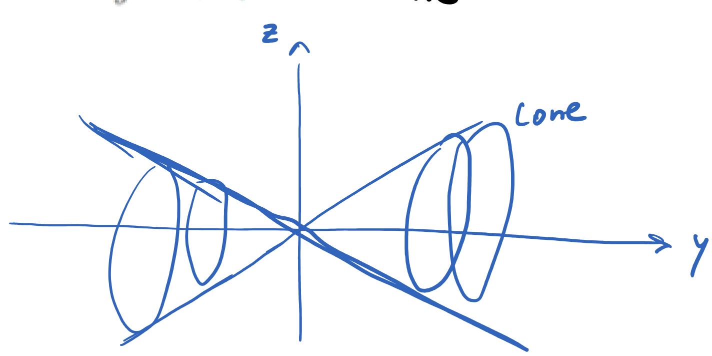


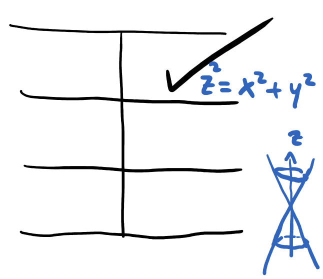
b) $z = -(x^2 + y^2)$



Elliptic Paraboloid

c) $y^2 = x^2 + z^2$ Cone





d) $x^2 + y^2 + z - 4x - 6y + 13 = 0.$

$x^2 - 4x + 4 + y^2 - 6y + 9 + z + 13 = 0 + 4 + 9$

$(x-2)^2 + (y-3)^2 + z = 0$

Let $x = x - 2$
$y = y - 3$

$x^2 + y^2 + z = 0$

$z = -(x^2 + y^2)$

Elliptic Paraboloid V: $(2, 3, 0)$