MATHEMATICS
TEXAS A&M UNIVERSITY

Instructor: Dr. Rosanna Pearlstein

Math Learning Center

Math 251 – Spring 2022
"Week-in-Review"

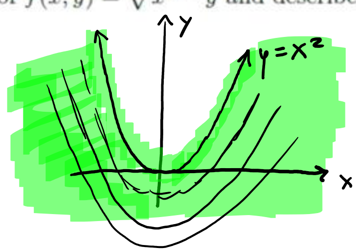
Wir 6: Exam 2 Review

Sections 14.1, 14.3-14.8

Problem 1. Sketch the domain of $f(x, y) = \sqrt{x^2 - y}$ and describe the level curves.

$x^2 - y \geq 0$

$y \leq x^2$



$\sqrt{x^2 - y} = k$

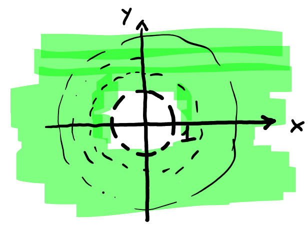
$x^2 - y = k^2$

$x^2 - k^2 = y$

Problem 2. Sketch the domain of $f(x, y) = \ln(y^2 + x^2 - 1)$ and describe the level curves.

$y^2 + x^2 - 1 > 0$

$x^2 + y^2 > 1$



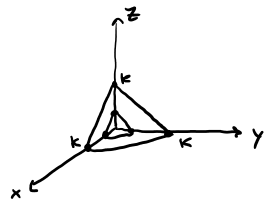
$\cancel{e}^{\cancel{\ln}(y^2 + x^2 - 1)} = e^k$

$x^2 + y^2 = e^k + 1 > 1$

Problem 3. What are the level surfaces to the equation $f(x, y, x) = x + y + z$?

$x + y + z = k$

planes



Problem 4. $f(x, y) = \sin(x^2 + y^2)$, find all first and second partial derivatives.

$\sin(x^2+y^2)$

$f_x = (2x)\cos(x^2+y^2)$

$f_{xx} = 2\cos(x^2+y^2) - 2x\,2x\sin(x^2+y^2)$

$f_{xy} = -2x\,2y\,\sin(x^2+y^2)$

$f_y = (2y)\cos(x^2+y^2)$

$f_{yx} = -2y\,2x\,\sin(x^2+y^2)$

$f_{yy} = 2\cos(x^2+y^2) - 2y\,2y\,\sin(x^2+y^2)$

$z_0 = f(1,1) = 2+1 = 3$

Problem 5. Find an equation for the tangent plane to the surface $z = 2x^2 + y^2$ at the point $P_0(1, 1)$

$$z - z_0 = f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0)$$

$$\boxed{z - 3 = 4(x-1) + 2(y-1)}$$

$f_x = 4x \to 4$

$f_y = 2y \to 2$

Problem 6. Find the tangent plane to the surface $\underbrace{2xy + 3yz + 7xz - 9}_{F} = 0$ at the point $\overset{P_0}{(1, 2, -1)}$.

$$F_x(P_0)\,(x - x_0) + F_y(P_0)\,(y - y_0) + F_z(P_0)(z - z_0) = 0$$

$$F_x = 2y + 7z \longrightarrow F_x(P_0) = 4 - 7 = -3$$

$$F_y = 2x + 3z \longrightarrow F_y(P_0) = 2 - 3 = -1$$

$$F_z = 3y + 7x \longrightarrow F_z(P_0) = 6 + 7 = 13$$

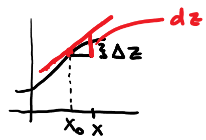
$$-3(x - 1) - 1(y - 2) + 13(z + 1) = 0$$

Problem 7. If $z = x^3y^2$, find the differential, dz, and explain what it measures.

$$dz = z_x\, dx + z_y\, dy$$

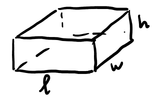
$$dz = 3x^2y^2\,dx + 2x^3y\,dy$$

$dz \approx \Delta z$



Problem 8. Consider a rectangular box with length l, width w and height h. If A is the surface area of the box, find the differential, dA.

$A = 2\ell w + 2hw + 2\ell h$



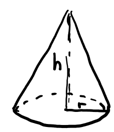
$dA = A_\ell \, d\ell + A_h \, dh + A_w \, dw$

$2(w+h)d\ell + 2(w+\ell)dh + 2(\ell+h)dw = dA.$ ✓

Problem 9. The height of a cone was measured to be 3 cm with a maximum error of 0.1 cm and the radius of the cone is measured to be 2 cm with a maximum error of 0.2 cm. Use differentials to estimate the maximum error in the calculated volume of the cone.

$r = 2$ cm $\quad dr = 0.2$ cm

$h = 3$ cm $\quad dh = 0.1$ cm



$$V = \frac{1}{3}\pi r^2 h$$

$$dV = V_r\, dr + V_h\, dh =$$

$$= \frac{\pi}{3}\underbrace{2rh}\, dr + \frac{\pi}{3} r^2\, dh$$

$$\frac{\pi}{3}\, 12\, \frac{2}{10} + \frac{\pi}{3}\, 4\, \frac{1}{10} = \frac{\pi}{3}\left(2.4 + 0.4\right) = \frac{\pi}{3} \cdot \frac{28}{10} = \frac{28\pi}{30} \text{ cm}^3$$

$$\frac{24+4}{10}$$

Problem 10. Use a linear approximation (tangent plane) to estimate $((2.1)^2 + (0.1)^3)^3$

$$L(x,y) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0)$$

$f(x,y) = (x^2+y^3)^3$ $\quad P_0\ (2,0) \quad f(P_0) = (2^2+0^3)^3 = 64.$

$f_x = 2x\,3(x^2+y^3)^2$
$6x\,(x^2+y^3)^2$
$12\ (4+0)^2 = 192$

$\begin{array}{r}12\\ 16\\ \hline 2\end{array}$

$f_y = 3y^2\,3(x^2+y^3)^2$
$9y^2\,\underbrace{(x^2+y^3)^2}$
0

$L(x,y) = 64 + 192(x-2) + 0\,(y-0)$
$64 + 192(2.1-2) \quad 0\,(0.1-0)$
$64 + 192\,\frac{1}{10} = 64 + 19.2 = \boxed{83.2}$

True value 85.824....

Problem 11. Use differentials to approximate $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$

$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$P_0(3, 2, 6)$

$f(P_0) = \sqrt{9+4+36} = 7$

$f_x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}$ $\frac{3}{7}$

$f_y = \frac{y}{\sqrt{}}$ $\frac{2}{7}$

$f_z = \frac{z}{\sqrt{}}$ $\frac{6}{7}$

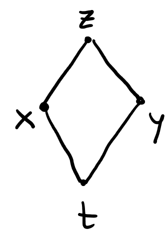
$L(x,y,z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0)$

$7 + \frac{3}{7}(3.02-3) + \frac{2}{7}(1.97-2) + \frac{6}{7}(5.99-6)$

$7 + \frac{3}{7}\frac{2}{100} - \frac{2}{7}\frac{3}{100} - \frac{6}{7}\frac{1}{100}$

$\frac{4900 + 6 - 6 - 6}{700} = \frac{4894}{700}$

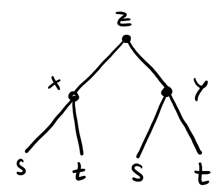
Problem 12. If $z = e^{x^2+y^2}$, $x = e^t$. $y = \cos t$, find $\frac{dz}{dt}$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$2x e^{x^2+y^2} e^t - 2y e^{x^2+y^2} \sin t.$$

Problem 13. For $z = xy$, $x = \underbrace{\cos(st^2)}$, $\underbrace{y = \sin(e^t)}$ find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$.



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

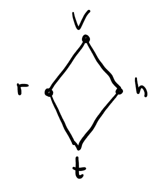
$$-(y\,2st)\sin(st^2) + x\,e^t\cos(e^t)$$

$$\frac{\partial z}{\partial s} = z_x\, x_s + z_y\, y_s$$

$$\underbrace{-y\,t^2\sin(st^2)} + x\cdot\cancel{0}$$

Problem 14. The radius and height of a circular cylinder change with time. When the height is 1 meter and the radius is 2 meters, the radius is increasing at a rate of 4 meters per second and the height is decreasing at a rate of 2 meters per second. At that same instant, find the rate at which the volume is changing.

$$\left.\frac{dV}{dt}\right|_{\substack{h=1,\ r=2 \\ \frac{dr}{dt}=4 \quad \frac{dh}{dt}=-2}} = ?$$



$$V = \pi r^2 h$$

$$\frac{dV}{dt} = V_r \frac{dr}{dt} + V_h \frac{dh}{dt}$$

$$(2\pi r h)(4) + (\pi r^2)(-2)$$

$$4\pi \cdot 4 + 4\pi(-2) = 16\pi - 8\pi = 8\pi \ \frac{m^3}{sec}$$

Problem 15. Let $f(x, y) = \sqrt{xy}$. Find the directional derivative of f at the point $P(4, 1)$ in the direction from P to $Q(6, 2)$

$$\boxed{D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u} \quad \text{as } |u| = 1}$$

$\vec{PQ} = Q - P = \langle 2, 1 \rangle = \vec{v}$

$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{4+1}} \langle 2, 1 \rangle$

$\vec{\nabla} f = \langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \rangle$

$\vec{\nabla} f(4,1) = \langle \frac{1}{4}, \frac{4}{4} \rangle$

$D_{\vec{u}} f(4,1) = \langle \frac{1}{4}, 1 \rangle \cdot \frac{1}{\sqrt{5}} \langle 2, 1 \rangle = \frac{1}{\sqrt{5}} \left(\frac{1}{2} + 1 \right) = \frac{1}{\sqrt{5}} \frac{3}{2} = \frac{3}{2\sqrt{5}}$

Problem 16. Let $f(x, y) = \sqrt{xy}$. What is the direction of the largest rate of change at the point $P(4, 1)$?

direction of $\vec{\nabla} f(P_0) = \langle \frac{1}{4}, 1 \rangle$

max rate of increase is $|\vec{\nabla} f(P_0)| =$

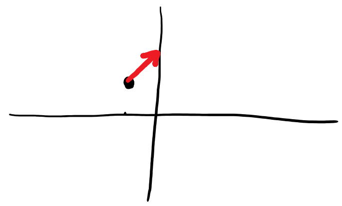
$= \sqrt{\frac{1}{16} + 1} = \sqrt{\frac{17}{16}}$

Problem 17. Let $f(x, y) = e^{x+y}$. What is the maximum rate of change at the point $P_0(-1, 1)$?

$$|\vec{\nabla} f(-1,1)|$$

$$\vec{\nabla} f\Big|_{P_0} = \langle e^{x+y}, e^{x+y} \rangle\Big|_{P_0} = \langle e^0, e^0 \rangle = \langle 1, 1 \rangle$$

max $\nearrow$ is $\sqrt{1+1} = \sqrt{2}$



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Problem 18. For the $f(x, y) = 2x^3 - xy^2 + 5x^2 + y^2 + 5$, find all local minima, maxima, and saddle points.

STEP 1: Critical pts. $\vec{\nabla} f = \vec{0}$ or dne

$\vec{\nabla} f = \langle 6x^2 - y^2 + 10x, -2xy + 2y \rangle = \langle 0, 0 \rangle$

$$\begin{cases} 6x^2 - y^2 + 10x = 0 \\ 2y(-x+1) = 0 \quad y=0 \text{ or } x=1 \end{cases}$$

If $y=0 \Rightarrow 6x^2 + 10x = 0 \Rightarrow x(6x+10) = 0$, $x=0$ $(0,0)$; $x = -\frac{10}{6} = -\frac{5}{3}$ $\left(-\frac{5}{3}, 0\right)$

If $x=1 \Rightarrow 6 - y^2 + 10 = 0 \Rightarrow y = \pm 4$ $(1,-4)$ $(1,4)$

STEP 2: Second derivative test

$$D = \begin{vmatrix} 12x+10 & -2y \\ -2y & -2x+2 \end{vmatrix}$$

$D(0,0) = \begin{vmatrix} \boxed{10} & 0 \\ 0 & 2 \end{vmatrix} = 20 > 0$ $f_{xx} = 10 > 0$ $(0,0)$ local min.

$-4 \cdot \frac{5}{3} + 10$

$D\left(-\frac{5}{3}, 0\right) = \begin{vmatrix} -10 & 0 \\ 0 & \frac{16}{3} \end{vmatrix} < 0 \Rightarrow \left(-\frac{5}{3}, 0\right)$ saddle pt.

$\frac{10}{3} + \frac{6}{3}$

$D(1,4) = \begin{vmatrix} 22 & -8 \\ -8 & 0 \end{vmatrix} = -64$ $(1,4)$ saddle pt.

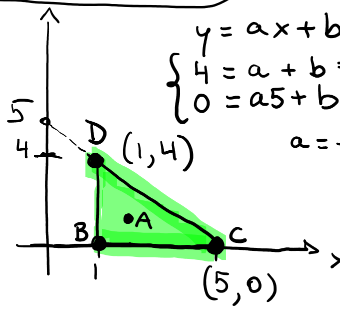
$D(1,-4) = \begin{vmatrix} 22 & 8 \\ 8 & 0 \end{vmatrix} = -64$ $(1,-4)$ saddle pt.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Problem 19. Find the absolute maximum and minimum values of $\boxed{f(x,y) = 7 + xy - x - 2y}$ over the closed triangular region with verticies $(1,0)$, $(5,0)$, $(1,4)$.

Step 1. $\vec{\nabla} f = \vec{0}$

$$\left.\begin{aligned} f_x &= y - 1 = 0 \\ f_y &= x - 2 = 0 \end{aligned}\right\} \text{ at } (2,1)\ A$$



$y = ax + b$

$$\begin{cases} 4 = a + b \Rightarrow 4 = a - 5a = -4a \\ 0 = a5 + b \Rightarrow b = -5a \end{cases}$$

$a = -1 \quad b = 5$

$\boxed{y = -x + 5}$

Step 2

$\overline{BC}$: $y = 0 \qquad 1 \le x \le 5$

$f(x) = 7 - x \qquad f(1)$

$$\left.\begin{matrix} (1,0) \\ (5,0) \end{matrix}\right\} \rightarrow \begin{aligned} f(1,0) &= 6 \\ f(5,0) &= 2 \end{aligned}$$

$\overline{BD}$: $x = 1 \qquad 0 \le y \le 4$

$f(y) = 7 + y - 1 - 2y$

$= 6 - y$

$$\left.\begin{matrix} (1,0) \\ (1,4) \end{matrix}\right\} \rightarrow \begin{aligned} f(1,0) &= 6 \\ f(1,4) &= +2 \end{aligned}$$

$f(2,1) = 7 + \cancel{2} - \cancel{2} - 2 = 5$

$f(3,2) = -9 + 18 - 3 = 6$

$\overline{CD}$: $\boxed{y = 5 - x} \qquad 1 \le x \le 5$

$f(x) = 7 + x(5 - x) - x - 2(5 - x)$

$7 + 5x - x^2 - x - 10 + 2x$

$f(x) = -x^2 + 6x - 3$

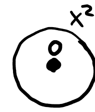
$f'(x) = -2x + 6$

$x = 3$

Step 3: Abs. max is 6 at $(1,0)$ & $(3,2)$
Abs. min is 2 at $(5,0)$ & $(1,4)$

Problem 20. Find the absolute maximum and minimum values of $f(x, y) = 2x^3 + y^4$ over the region $D = \{(x, y) : x^2 + y^2 \leq 1\}$

1. $\vec{\nabla} f = \vec{0}$ $\quad \langle 6x^2, 4y^3 \rangle = \langle 0, 0 \rangle$
at $(0, 0)$



$x^2+y^2=1 \to g$

2. Lagrange multipliers:
$\vec{\nabla} f = \lambda \vec{\nabla} g$

$f(0) = 0$

$$\begin{cases} \langle 6x^2, 4y^3 \rangle = \lambda \langle 2x, 2y \rangle \\ x^2 + y^2 = 1 \end{cases}$$

If $x = 0 \Rightarrow 0^2 + y^2 = 1$ $\quad (0, \pm 1)$
If $y = 0 \Rightarrow x^2 + 0^2 = 1$ $\quad (\pm 1, 0)$

$$\begin{cases} 6x^2 = 2\lambda x & \Rightarrow \boxed{3x = \lambda} \\ 4y^3 = 2\lambda y & \Rightarrow \boxed{2y^2 = \lambda} \\ x^2 + y^2 = 1 \end{cases} \Rightarrow 3x = 2y^2 \Rightarrow x = \frac{2}{3} y^2$$

$$\frac{4}{9} y^4 + y^2 - 1 = 0$$

$y^2 = z$

$\frac{4}{9} z^2 + z - 1 = 0$

....

$$4z^2 + 9z - 9 = 0$$

$$y^2 = \frac{-9 \pm \sqrt{81 - 4 \cdot 4 \cdot 9}}{8} = \frac{-9 \pm 15}{8}$$

~~negative~~

$\frac{6}{8} = \frac{3}{4}$

$$y = \pm \frac{\sqrt{3}}{2}$$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \to$ function $= \frac{2}{8} + \frac{9}{16} = \frac{13}{16}$

$$\underbrace{(0, 1) \quad (0, -1)}_{f = 1} \quad \underbrace{(1, 0)}_{f = 2} \quad \underbrace{(-1, 0)}_{f = -2}$$

$f = 2$ Absolute max

$f = -2$ Absolute min.

$f = 1$ $f=2$ Absolute max $f=-2$ Absolute min.

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Problem 21. Use the method of Lagrange to find the maximum and minimum values of $\underbrace{f(x, y) = 6x + 6y}$ subject to the constraint $\underbrace{x^2 + y^2 = 18}_{g}$.

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ x^2 + y^2 = 18 \end{cases}$$

$\langle 6, 6 \rangle = \lambda \langle 2x, 2y \rangle$

$6 = 2\lambda x$
$6 = 2\lambda y$

$x = y$

$x^2 + x^2 = 18$

$2x^2 = 18$

$x^2 = 9$

$x = \pm 3$

$\underbrace{(3, 3)}_{\text{absolute max value } = 18+18=36}$ $\underbrace{(-3, -3)}_{\text{absolute min value } = -36}$

Problem 22. Use the method of Lagrange to find the maximum and minimum values of $f(x, y) = y^2 - x^2$ subject to the constraint $\frac{1}{4}x^2 + y^2 = 25$ $\circledast$

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$$\begin{cases} \langle -2x, 2y \rangle = \lambda \langle \frac{1}{2}x, 2y \rangle \\ \circledast \end{cases}$$

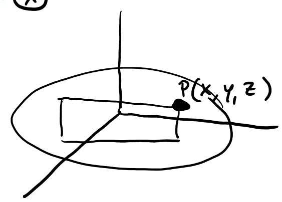
$-2x = \lambda \frac{x}{2} \Rightarrow x\left(-2 - \frac{\lambda}{2}\right) = 0$

$2y = \lambda 2y \Rightarrow y(2 - 2\lambda) = 0$ < $y = 0$ OR $\lambda = 1 \Rightarrow x\left(-2 - \frac{1}{2}\right) = 0 \Rightarrow x = 0$

If $y = 0 \Rightarrow \frac{1}{4}x^2 + 0^2 = 25 \Rightarrow x^2 = 100 \Rightarrow (\pm 10, 0)$ $f = -100$ abs. min. [redacted]

If $x = 0 \Rightarrow 0^2 + y^2 = 25$ $(0, \pm 5)$ $f = 25$ abs. max

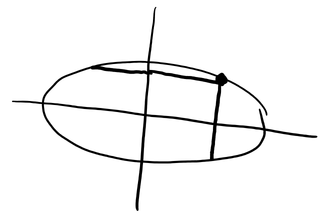
Problem 23. Find the volume of the largest rectangular box with faces parallel to the coordinate planes than can be inserted in the ellipsoid $\underbrace{16x^2 + 4y^2 + 9z^2 = 144}_{g}$. (*)



$h = 2z$
$w = 2x$
$\ell = 2y$

$V = 8xyz$

$\vec{\nabla} V = \lambda \vec{\nabla} g$



$\langle 8yz, 8xz, 8xy \rangle = \lambda^{\neq 0} \langle 32x, 8y, 18z \rangle$

(*)

$$\begin{cases} x \quad 8yz = 32\lambda x^2 \\ y \quad 8xz = 8\lambda y^2 \\ z \quad 8xy = 18\lambda z^2 \end{cases}$$

(*)

$32\cancel{\lambda} x^2 = 8\cancel{\lambda} y^2 = 18\cancel{\lambda} z^2$

$y^2 = x^2 \frac{32}{8} = 4x^2 \Rightarrow y = 2x$

$y^2 = \frac{\cancel{18}^{9}}{\cancel{8}4} z^2 \Rightarrow y = \frac{3}{2} z$

$x = \frac{1}{2} y$
$z = \frac{2}{3} y$

$\cancel{16}^{4}\left(\frac{1}{\cancel{4}} y^2\right) + 4y^2 + \cancel{9}\left(\frac{4}{\cancel{9}} y^2\right) = 144$

$(4 + 4 + 4)\, y^2 = 144$

$y^2 = 12 \qquad y = \sqrt{12} = 2\sqrt{3}$

$\boxed{x = \sqrt{3}, \ y = 2\sqrt{3}, \ z = \frac{4}{3}\sqrt{3}}$ $\max Vol = 8(\sqrt{3})(2\sqrt{3})\left(\frac{4}{3}\sqrt{3}\right).$