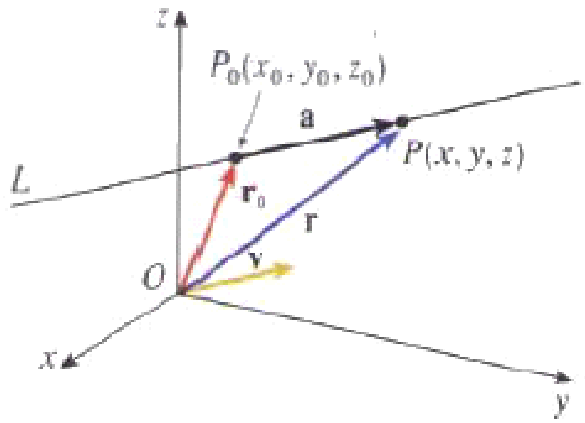
Section 12.5: Equations of Lines and Planes

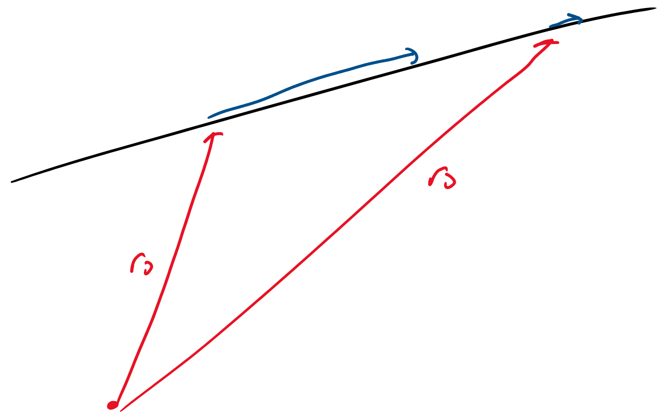
$y = 3x + 2$

Definition: The vector equation of a line is found by the formula

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}_0$ is a vector representation of a point on the line, $\mathbf{v}$ is a directional vector of the line (i.e. a vector that is parallel to the line), and $t \in \Re$.





Example: Find the vector equation and the parametric equations of a line though the point $(1, 2, 3)$ where the line is parallel to the vector $\mathbf{v} = < 2, 5, 10 >$.

point ✓

direction vector. ✓

$r_0 = \langle 1, 2, 3 \rangle$

$v = \langle 2, 5, 10 \rangle$

vector eq.

$r(t) = \langle 1, 2, 3 \rangle + t \langle 2, 5, 10 \rangle$

$r(t) = \langle 1+2t, 2+5t, 3+10t \rangle$

parametric equations

$x = 1+2t$

$y = 2+5t$

$z = 3+10t$

$t =$ any #

A

B Example: Find the vector equation of the line through the points $(3, 5, 5)$ and $(2, 1, -5)$. Also give the parametric equations of this line. Where does the line intersect the xy-plane?

point. ✓

$r_0 = \langle 3, 5, 5 \rangle$

direction vector: $v = \overrightarrow{BA} = \langle 1, 4, 10 \rangle$

vector eq:

$r(t) = \langle 3, 5, 5 \rangle + t \langle 1, 4, 10 \rangle$

$= \langle 3+t, 5+4t, 5+10t \rangle$

parametric eq.

$x = 3+t$

$y = 5+4t$

$z = 5+10t$

Intersect xy plane is $z=0$

$0 = 5+10t$

$-5 = 10t$

$t = -\frac{1}{2}$

$x = 3 - \frac{1}{2} = 2.5$

$y = 5 + 4\left(\frac{-1}{2}\right) = 5 - 2 = 3$

$z = 0$

point $(2.5, 3, 0)$

Example: Is the point $(7, 10, 17)$ on the line $\mathbf{r} =< 1 + 3t,\ 2 + 4t,\ 3 + 7t >$?

$7 = 1 + 3t$

$6 = 3t$

$2 = t$

$r(2) = < 7, 10, 17 >$ ✓

yes

$x = x_0 + at$
$y = y_0 + bt$
$z = z_0 + ct$

Symmetric equations of a line: If a, b, c $\neq 0$ and line L goes through the point (x_0, y_0, z_0) with directional vector $\langle a, b, c\rangle$ then

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

If, for example, $a = 0$ then the symmetric equations have the form:

$$x = x_0,\ \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

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new point.

Example: Find the symmetric equations of the line through the point $(5, 8, -2)$ and parallel to the line

$x = 2 + 4t$
$y = 3 + 2t$
$z = 1 + 6t$

$v = \langle 4, 2, 6 \rangle$

parametric eq.

$x = 5 + 4t$
$y = 8 + 2t$
$z = -2 + 6t$

$$\boxed{\frac{x-5}{4} = \frac{y-8}{2} = \frac{z+2}{6}}$$

$\frac{1-z}{2} = m$ $\quad$ $1-z = 2m$

$1-2m = z$

Definition: **Skew** lines are lines that are not parallel and do not intersect.

Example: Are these lines parallel, skew, or intersecting? If intersecting, find the point of intersection.

$$L_1: \quad \frac{x+2}{3} = \frac{y-5}{-4} = \frac{1-z}{2} = m \longrightarrow \begin{aligned} x &= -2+3m \\ y &= 5-4m \\ z &= 1-2m \end{aligned}$$

$V_1 = \langle 3, -4, -2 \rangle$

and

$$L_2: \quad x = 1-t, \quad y = 3+2t, \quad z = -12-3t$$

$V_2 = \langle -1, 2, -3 \rangle$

parallel? NO!

Intersecting

$1-t = 3m-2$

$-t = 3m-2-1$

$-t = 3m-3$

$t = -3m+3$

$3+2t = 5-4m$

$3+2(-3m+3) = 5-4m$

$3-6m+6 = 5-4m$

$9-5 = 2m$

$4 = 2m$

$2 = m$

$t = -3(2)+3$

$t = -6+3$

$t = -3$

$-12-3t = 1-2m$

Line 2

$z = -12-3t$

$z = -12-3(-3)$

$= -12+9$

$= -3$ ✓

Line 1

$z = 1-2m$

$= 1-2(2)$

$z = 1-4 = -3$ ✓

Lines Intersect

find point

Line 1 $\quad m = 2$

$x = 3(2)-2 = 4$

$y = 5-4(2) = 5-8 = -3$

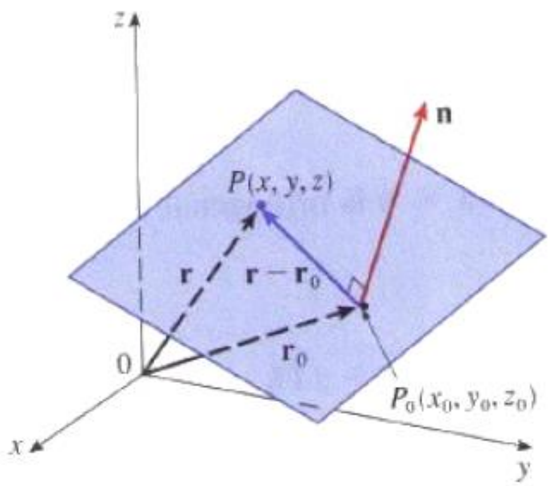
$z = 1-2(2) = -3$

point $(4, -3, -3)$

Line 2 $\quad t = -3$ value.

Pg 7: planes

A **plane** is determined by a point $P_0(x_0, y_0, z_0)$ and a vector $\mathbf{n} =< a, b, c >$ that is orthogonal to the plane. The vector $\mathbf{n}$ is called a normal vector.



Let $P(x,y,z)$ any point on the plane.

$r = \langle x, y, z \rangle \qquad r_0 = \langle x_0, y_0, z_0 \rangle$

$r - r_0 = \langle x - x_0, y - y_0, z - z_0 \rangle$

Then

Vector equation of the plane: $n \cdot (r - r_0) = 0$

$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

Scalar equation of the plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$ax + by + cz = ax_0 + by_0 + cz_0$

Example: Find an equation of the plane through the point $(1, 2, 3)$ and is orthogonal to the vector $\langle 3, 4, 7 \rangle$

point ✓

normal vector. ✓✓

$$3(x-1) + 4(y-2) + 7(z-3) = 0$$

$$3x + 4y + 7z = 3(1) + 4(2) + 7(3)$$

$$\begin{array}{r} 3 \\ 8 \\ 21 \\ \hline 32 \end{array}$$

$$3x + 4y + 7z = 32$$

Example: Find an equation of the plane through the points $A(1, 1, 3)$, $B(-1, 3, 2)$, and $C(1, -1, 2)$.

point ✓

normal vector ✗

$\vec{AB} = \langle -2, 2, -1 \rangle$

$\vec{AC} = \langle 0, -2, -1 \rangle$

$n = \vec{AB} \times \vec{AC} = \dots = \langle -4, -2, 4 \rangle$

$$-4(x-1) - 2(y-1) + 4(z-3) = 0$$

$$-4x - 2y + 4z = 6$$

Example Find an equation of the plane through the point $(1, 2, 3)$ and contains the line $x = 2 + 4t$, $y = 1 + 5t$, $z = -1 + 3t$

A

plane

point ✓

normal vector

line

point

B $(2, 1, -1)$

$V = \langle 4, 5, 3 \rangle$ direction vector

$\overrightarrow{AB} = \langle 1, -1, -4 \rangle = \langle 2-1, 1-2, -1-3 \rangle$

normal vector $n = \overrightarrow{AB} \times V = \dots = \langle 17, -19, 9 \rangle$

$$\boxed{17(x-1) - 19(y-2) + 9(z-3) = 0}$$

$$17x - 19y + 9z = 17(1) - 19(2) + 9(3)$$

$$\boxed{17x - 19y + 9z = 6}$$

Example: You are given two different lines. Does there exist a plane that contains the given lines? If not, what conditions are needed so that there is a plane that contains the given lines?

either parallel lines or the lines Intersect.

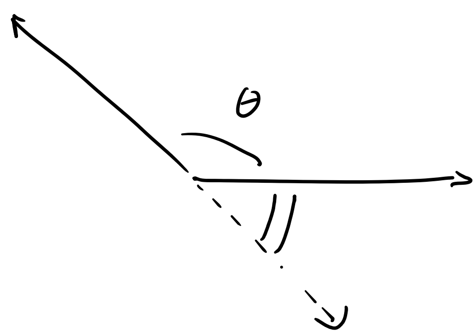
Skew lines can not be in the same plane.

Definition: Two planes are parallel if their normal vectors are parallel.

Definition: Two planes are perpendicular(orthogonal) if their normal vectors are perpendicular.

Definition: The angle between two non-parallel planes is the acute angle between the normal vectors.

$0 \le \theta \le \frac{\pi}{2}$



Example: Determine if the pairs of are parallel, orthogonal, or neither?

P_1 : $4x + 2y - 8z = 15$

P_2 : $2x + y - 4z = 12$

P_3 : $3x + 2y + 2z = 10$

$n_1 = \langle 4, 2, -8 \rangle$

$n_2 = \langle 2, 1, -4 \rangle$

$n_3 = \langle 3, 2, 2 \rangle$

n_1 + n_2 are parallel
So P_1 + P_2 are parallel

perp

$$n_2 \cdot n_3 = 2(3) + 1(2) + (-4)(2)$$
$$= 6 + 2 + -8$$
$$= 0$$

n_2 + n_3 are perp.

Thus n_1 + n_3 are perp

So P_1 + P_2 are Both perp to P_3

Example: Find an equation of the line of intersection, L, of these two planes.

$x - y + 3z = 0$ $\quad n_1 = \langle 1, -1, 3 \rangle$

$x + y + 4z = 2$ $\quad n_2 = \langle 1, 1, 4 \rangle$

line

point.
direction vector

find a point

let $z=0$

$$x - y = 0$$
$$x + y = 2$$
$$2x = 2$$
$$x = 1 \qquad y = 1$$

point A $(1, 1, 0)$

let $z=1$

$$x - y + 3 = 0 \rightarrow x - y = -3$$
$$x + y + 4 = 2 \rightarrow x + y = -2$$
$$2x = -5$$
$$x = \frac{-5}{2} = -2.5$$

$$-2.5 + y = -2$$
$$y = .5$$

point B $(-2.5, .5, 1)$

find direction vector

$$\overrightarrow{AB} = \langle -3.5, -.5, 1 \rangle$$

Line:

$$x = 1 - 3.5t$$
$$y = 1 - .5t$$
$$z = 0 + t$$

Example: Find an equation of the line of intersection, L, of these two planes.

$x - y + 3z = 0$ $\quad n_1 = \langle 1, -1, 3 \rangle$

$x + y + 4z = 2$ $\quad n_2 = \langle 1, 1, 4 \rangle$

Still need a point.
$(1, 1, 0)$

$V = n_1 \times n_2 = \dots = \langle -7, -1, 2 \rangle$

line

$x = 1 - 7t$

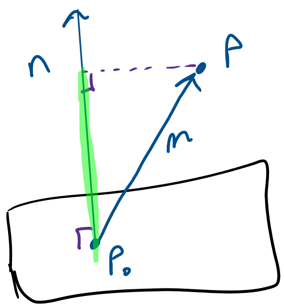
$y = 1 - t$

$z = 0 + 2t$

The distance between a point $P\ (x, y, z)$ to the plane $\underbrace{ax + by + cz + d = 0}$ is $\longrightarrow ax + by + cz = -d$

Let $P_0(x_0, y_0, z_0)$ be any point in the plane. $\longrightarrow a x_0 + b y_0 + c z_0 = -d$

Let $m = \overrightarrow{P_0 P} = \langle x - x_0, y - y_0, z - z_0 \rangle$



normal vector $n = \langle a, b, c \rangle$

$$\text{distance} = \left| \text{comp}_n m \right| = \left| \frac{m \cdot n}{|n|} \right|$$

$$= \left| \frac{a(x - x_0) + b(y - y_0) + c(z - z_0)}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$= \frac{\left| ax + by + cz - \overbrace{(ax_0 + by_0 + cz_0)}^{-d} \right|}{\sqrt{a^2 + b^2 + c^2}}$$

$$\boxed{\text{distance} = \frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}} \quad \text{where } ax + by + cz + d = 0}$$

Example: Find the distance between the point $(3, -2, 7)$ and the plane $\underbrace{4x - 6y + z = 5}$

$n = \langle 4, -6, 1 \rangle$

$4x - 6y + z - 5 = 0$

$$\text{distance} = \frac{|4(3) - 6(-2) + (7) - 5|}{\sqrt{4^2 + (-6)^2 + 1^2}}$$

$$= \frac{|12 + 12 + 7 - 5|}{\sqrt{16 + 36 + 1}} = \frac{26}{\sqrt{53}}$$