Section 13.3: Arc Length and Curvature

In Cal II, the arc length of a two-dimensional smooth curve that is only traversed once on an interval I was given by

$$L = \int_I ds \text{ or } L = \int_I \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt$$

$ds = \sqrt{(x')^2 + (y')^2}\ dt$

This can be extended to a space curve. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ on the interval $a \le t \le b$, then the length of the curve is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt \qquad \text{or} \qquad L = \int_a^b |\mathbf{r}'(t)|\ dt. = \int_I ds$$

$ds = |r'(t)|\ dt$

$ds = \sqrt{(x')^2 + (y')^2 + (z')^2}\ dt$

Note: A curve $\mathbf{r}(t)$ is called smooth on an interval if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on the interval. A smooth curve has no sharp corners or cusps, i.e. the tangent vector has continuous movement. The arc length formula holds for smooth and piecewise-smooth curves.

$6\pi = 3t$

$t = 2\pi$

Example: Find the length of the arc for $\mathbf{r}(t) = \langle 3t, 2\sin(t), 2\cos(t)\rangle$ from the point $\underbrace{(0,0,2)}_{\substack{t=0\\ \text{start}}}$ to $\underbrace{(6\pi,0,2)}_{\substack{t=2\pi\\ \text{end.}}}$.

$$r' = \langle 3, 2\cos(t), -2\sin(t) \rangle$$

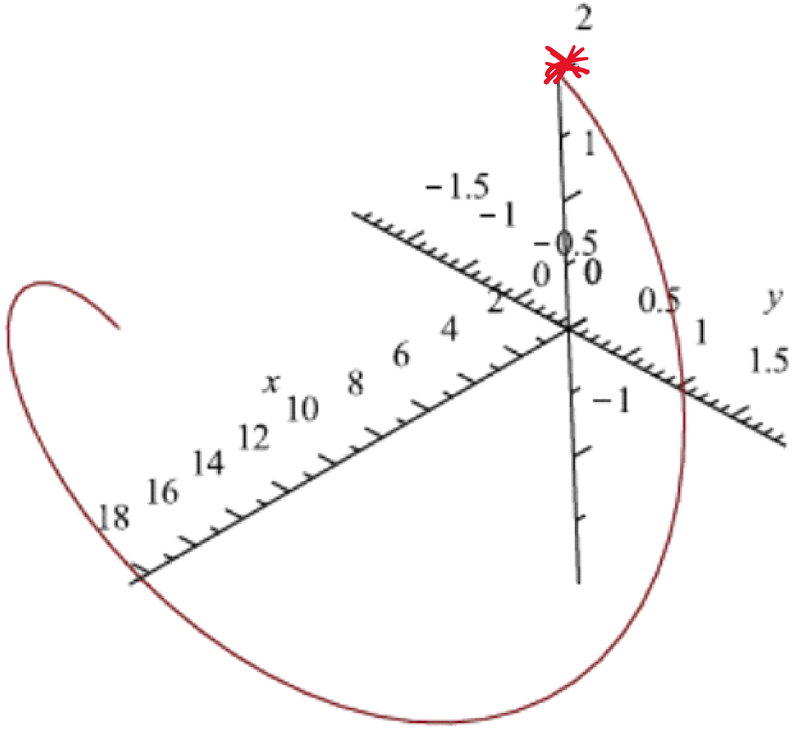
$$L = \int_I ds = \int_0^{2\pi} \sqrt{3^2 + (2\cos t)^2 + (-2\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9 + 4\cos^2 t + 4\sin^2 t}\, dt = \int_0^{2\pi} \sqrt{9 + 4[\cos^2 t + \sin^2 t]}\, dt$$

$$= \int_0^{2\pi} \sqrt{9+4}\, dt = \int_0^{2\pi} \sqrt{13}\, dt = t\sqrt{13}\Big|_0^{2\pi}$$

$$= 2\pi\sqrt{13} - 0 = 2\pi\sqrt{13}$$

graph



$$\frac{ds}{dt} = |r'(t)| \cdot 1 = |r'(t)|$$

Definition: The **arc length function**, s, is $s(t) = \int_a^t |\mathbf{r}'(u)|\, du$.

The arc length s is called the **arc length parameter**.

start

Example: Find the arc length function for $\mathbf{r}(t) = \langle 1, t^2, t^3 \rangle$ from the point $\underbrace{(1,0,0)}_{\to\, t=0}$ in the direction of increasing t.

$$r' = \langle 0, 2t, 3t^2 \rangle$$

$$r' = \langle 0, 2u, 3u^2 \rangle$$

$$s = \int_0^t \sqrt{0 + (2u)^2 + (3u^2)^2}\, du = \int_0^t \sqrt{4u^2 + 9u^4}\, du$$

$$= \int_0^t \sqrt{u^2(4+9u^2)}\, du = \int_0^t u\sqrt{4+9u^2}\, du$$

Substitution

$A = 4 + 9u^2$

$dA = 18u\, du$

$\frac{1}{18u}\, dA = du$

$$= \int_{u=0}^{u=t} u\sqrt{A}\, \frac{1}{18u}\, dA = \int_{u=0}^{u=t} \frac{1}{18} A^{1/2}\, dA$$

$$= \frac{1}{18} A^{3/2} \cdot \frac{2}{3} \Big|_{u=0}^{u=t} = \frac{1}{27}(4+9u^2)^{3/2} \Big|_0^t$$

$$\boxed{s = \frac{1}{27}(4+9t^2)^{3/2} - \frac{1}{27}(4)^{3/2}}$$

Example: Reparametrize the curve $\mathbf{r}(t) = \langle 1+2t,\ 3+t,\ -5t\rangle$ with respect to arc length measured from the point where $t=0$ in the direction of increasing t.

$r'(t) = \langle 2, 1, -5 \rangle$

step 1

$$s = \int_0^t \sqrt{2^2 + 1^2 + (-5)^2}\, du = \int_0^t \sqrt{4+1+25}\, du = \int_0^t \sqrt{30}\, du$$

$$= u\sqrt{30}\Big|_0^t = t\sqrt{30}$$

$$s = t\sqrt{30}$$

Step 2) solve for t.

$$t = \frac{s}{\sqrt{30}}$$

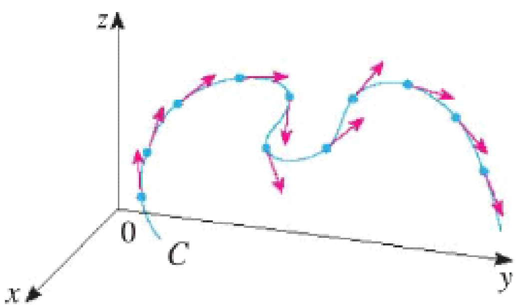
Step 3) give answer.

$$r(s) = \left\langle 1 + \frac{2s}{\sqrt{30}},\ 3 + \frac{s}{\sqrt{30}},\ \frac{-5s}{\sqrt{30}} \right\rangle$$

Pg 5: curvature

Definition: The **curvature**, κ, of a curve is defined to be the magnitude of the rate of change of the unit tangent vector with respect to the arc length is given by $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$

$$\left|\frac{dT}{ds}\right| = \frac{\left|\frac{dT}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \frac{|T'(t)|}{|r'(t)|}$$



Unit tangent vectors at equally spaced points on C

Theorem The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = |\mathbf{r}''(s)|$$

Example: Find the curvature of $\mathbf{r}(t) = \left\langle -\sqrt{2}\sin t,\ \cos t,\ \cos t \right\rangle$.

$$r' = \langle -\sqrt{2}\cos t,\ -\sin t,\ -\sin t \rangle$$

$$|r'| = \sqrt{2\cos^2 t + \sin^2 t + \sin^2 t} = \sqrt{2\cos^2 t + 2\sin^2 t} = \sqrt{2}$$

$$T = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}} \langle -\sqrt{2}\cos t,\ -\sin t,\ -\sin t \rangle$$

$$T' = \frac{1}{\sqrt{2}} \langle \sqrt{2}\sin t,\ -\cos t,\ -\cos t \rangle = \left\langle \sin t,\ -\frac{\cos t}{\sqrt{2}},\ -\frac{\cos t}{\sqrt{2}} \right\rangle$$

$$|T'| = \sqrt{\sin^2 t + \frac{1}{2}\cos^2 t + \frac{1}{2}\cos^2 t} = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1$$

$$\kappa = \frac{|T'(t)|}{|r'(t)|} = \frac{1}{\sqrt{2}}$$

Example: Find the curvature of $\mathbf{r}(t) = \left\langle 1+t, 1-t, 3t^2 \right\rangle$

$r' = \langle 1, -1, 6t \rangle$

$r'' = \langle 0, 0, 6 \rangle$

$r' \times r'' = \dots = \langle -6, -6, 0 \rangle$

$|r' \times r''| = \sqrt{36+36+0}$
$= \sqrt{72}$

$\kappa = \dfrac{\sqrt{72}}{\left(\sqrt{2+36t^2}\right)^3}$

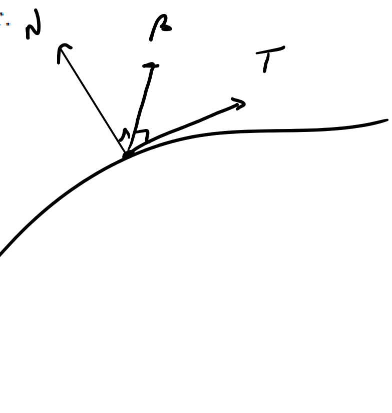
$|r'| = \sqrt{1+1+36t^2} = \sqrt{2+36t^2}$

$T = \dfrac{r'(t)}{|r'(t)|} = \dfrac{1}{\sqrt{2+36t^2}} \langle 1, -1, 6t \rangle$

derivative is a product Rule
i.e. complicated.

$$T = \frac{r'(t)}{|r'(t)|}$$

Note: The **unit normal vector** is defined as $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ and the **binormal vector** is defined as $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$. $\mathbf{B}(t)$ is also a unit vector.



The plane determined by $\mathbf{N}(t)$ and $\mathbf{B}(t)$ is called the **normal plane**.
The plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$ is called the **osculating plane**.