Section 14.3: Partial Derivatives

Here is a chart that gives the heat index, $f(T,H)$, as a function of actual Temperature (T) and relative humidity(H).
The heat index when the actual temperature is 96^oF and the relative humidity is 70% is 125^oF, i.e. $f(96,70) = 125^oF$.
What is the rate of change of the heat index when the actual temperature is 96^oF and the relative humidity is 70%?

Relative humidity (%)

Actual temperature (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Relative Humidity held fixed: $H = 70\%$

average rate of change from $T = 94$ to $T = 96$ is $\dfrac{125 - 118}{96 - 94} = 3.5^o$F per degree(actual temp)

average rate of change from $T = 96$ to $T = 98$ is $\dfrac{133 - 125}{98 - 96} = 4$

Actual temperature held fixed: $T = 96^oF$

average rate of change from $H = 65$ to $H = 70$ is $\dfrac{125 - 121}{70 - 65} = .8^o$ F per %

average rate of change from $H = 70$ to $H = 75$ is $\dfrac{130 - 125}{75 - 70} = 1$

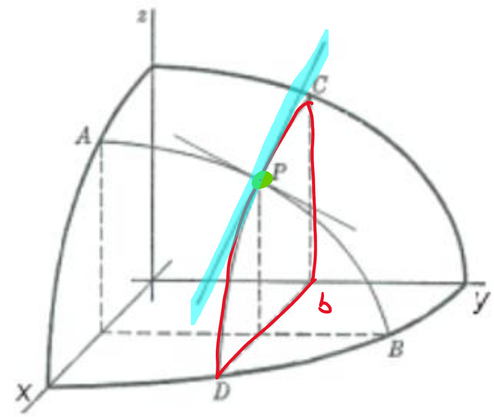
Definition: If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Geometric Interpretation of Partial Derivatives:

- $f_x(a,b)$ is the slope of the trace where the plane $y = b$ intersects the graph of $z = f(x,y)$ at the point (a,b).

- $f_y(a,b)$ is the slope of the trace where the plane $x = a$ intersects the graph of $z = f(x,y)$ at the point (a,b).



$\frac{df}{dx}$ $\frac{\partial f}{\partial x}$

Notations for Partial Derivatives: The alternate notations for the partial derivative of $z = f(x, y)$ with respect to x are

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_x f = D_1 f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_y f = D_2 f$$

$4/16$
8
128

Example: If $f(x, y) = x^3 + 3y^2 + 4x^2y^4$, find $f_x(1, 2)$ and $f_y(1, 2)$.

$f_x = 3x^2 + 0 + 8xy^4$

$f_x = 3x^2 + 8xy^4$

$f_x(1,2) = 3(1)^2 + 8(1)(2)^4$

$= 3 + 8(16)$

$= 3 + 128$

$f_x(1,2) = 131$

If you go from $(1,2)$ to $(2,2)$ the function increased by approx. 131

$f_y = 0 + 6y + 16x^2y^3$

$f_y = 6y + 16x^2y^3$

$f_y(1,2) = 6(2) + 16(1)^2(2)^3$

$= 12 + 16(8)$

$= 12 + 128$

$f_y(1,2) = 140$

If you go from $(1,2)$ to $(1,3)$ $f(x,y)$ inc. by approx 140.

Example: Find all of the first order partial derivatives for

$g(x, y, z) = x^2 \tan(4x + z^3) + y^5$.

$$\frac{\partial g}{\partial x} = g_x = 2x \tan(4x + z^3) + x^2 \cdot \sec^2(4x + z^3) \cdot 4 + 0$$

$$g_y = 0 + 5y^4$$

$$g_z = x^2 \sec^2(4x + z^3) \cdot (3z^2) + 0$$

$x^2 + e^y + \tan(y) + x^3 y = y + 2x$

$z = f(x,y)$

Example: Find $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y with the equation

$x^2 + y^3 + z^4 + 5xyz = 5$

$\frac{\partial z}{\partial y} = z_y$

product Rule

$$\frac{\partial}{\partial y}\left(x^2 + y^3 + z^4 + 5xyz\right) = \frac{\partial}{\partial y} 5$$

$$0 + 3y^2 + 4z^3 \frac{\partial z}{\partial y} + 5xz + 5xy \cdot \frac{\partial z}{\partial y} = 0$$

$$4z^3 \frac{\partial z}{\partial y} + 5xy \frac{\partial z}{\partial y} = -3y^2 - 5xz$$

$$\left(4z^3 + 5xy\right)\frac{\partial z}{\partial y} = -3y^2 - 5xz$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-3y^2 - 5xz}{4z^3 + 5xy}}$$

Higher Derivatives: Since $z = f(x, y)$ is a function of two variables, then its partial derivatives(first order), f_x and f_y, are also functions of two variables. Thus we can take partial derivatives of the of the first order partials. This gives **second order partial derivatives.**

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\underbrace{\partial y}_{2^{nd}}\underbrace{\partial x}_{1^{st}}} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

Example: Find the second partial derivatives of $f(x, y) = x^3e^{2y} + x^5y^3 + 2$

$f_x = 3x^2e^{2y} + 5x^4y^3 + 0$

$f_x = 3x^2e^{2y} + 5x^4y^3$

$f_y = x^3 \cdot 2e^{2y} + x^5 \cdot 3y^2 + 0$

$f_y = 2x^3e^{2y} + 3x^5y^2$

$f_{xx} = 6xe^{2y} + 20x^3y^3$

$f_{xy} = 3x^2 \cdot 2e^{2y} + 5x^4 \cdot 3y^2$

$f_{xy} = 6x^2e^{2y} + 15x^4y^2$

$f_{yy} = 2x^3 \cdot 2e^{2y} + 3x^5 \cdot 2y$

$= 4x^3e^{2y} + 6x^5y$

$f_{yx} = 6x^2e^{2y} + 15x^4y^2$

Example: Find the second partial derivatives of $f(x, y) = \ln(x^2 + y^2 + 1)$

$$f_x = \frac{2x}{x^2+y^2+1}$$

$$f_y = \frac{2y}{x^2+y^2+1}$$

$$f_{xx} = \frac{(x^2+y^2+1)\cdot 2 - 2x(2x)}{(x^2+y^2+1)^2}$$

$$= \frac{2x^2+2y^2+2-4x^2}{(x^2+y^2+1)^2}$$

$$= \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2}$$

$$f_{yy} = \frac{(x^2+y^2+1)2 - 2y(2y)}{(x^2+y^2+1)^2}$$

$$= \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2}$$

$$f_{xy} = \frac{(x^2+y^2+1)(0) - 2x(2y)}{(x^2+y^2+1)^2}$$

$$= \frac{-4xy}{(x^2+y^2+1)^2}$$

$$f_{yx} = \frac{(x^2+y^2+1)(0) - 2y\cdot 2x}{(x^2+y^2+1)^2}$$

$$= \frac{-4xy}{(x^2+y^2+1)^2}$$

Clairaut's Theorem. Suppose f is defined on a disk D that contains the point (a, b). IF the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Using Clairaut's Theorem it can be shown that $f_{xyx} = f_{xxy} = f_{yxx}$ if these functions are continuous.

Example: Find f_{xy} for $f(x, y, z) = x^2yz + x^5\left(x^2 + z^3\right)^6$. ← $f(x,y)$ is cont.

by Clairaut's Thm $f_{xy} = f_{yx}$

$f_y = x^2 z$

$f_{yx} = 2xz = f_{xy}$