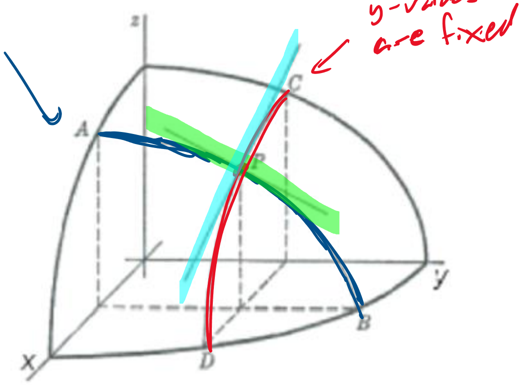
Definition: Suppose the surface S has the equation $z = f(x, y)$, where f has continuous first partials, and let $P(a, b, c)$ be a point on the surface. If C_1 and C_2 be the curves obtained by intersecting the planes $x = a$ and $y = b$ with the surface, then T_1 and T_2 are the respective tangent lines to the curves at point P. The **tangent plane** to the surface S at the point $P(a, b, c)$ is defined to be the plane that contains both of the tangent lines at point P.

xvalues are fixed

y-values are fixed



$V_1 = \langle 0, 1, f_y(a,b) \rangle$

$V_2 = \langle 1, 0, f_x(a,b) \rangle$

$$n = V_2 \times V_1 = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix}$$

$$\begin{cases} n = \langle -f_x(a,b), -f_y(a,b), 1 \rangle \\ n = \langle -f_x, -f_y, 1 \rangle \end{cases}$$

also $\langle f_x, f_y, -1 \rangle$

Theorem: An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(a, b, c)$ or $P(a, b, f(a, b))$ is

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + 1(z-c) = 0$$

The normal vector for this plane is

The **normal line** is the line though the point $P(a, b, c)$ that is perpendicular to the tangent plane.

$$x = a - f_x(a,b) \cdot t$$
$$y = b - f_y(a,b) \cdot t$$
$$z = c + t$$

$z = f(x,y)$

Example: Find an equation of the tangent plane to the graph of the function $z = 3x^2 + y^4$ at the point $(2, 1, 13)$.

$n = \langle -f_x, -f_y, 1 \rangle$
$= \langle -z_x, -z_y, 1 \rangle$

$z_x = f_x = 6x$ $\qquad f_x(2,1) = 12$

$n = \langle -12, -4, 1 \rangle$

$z_y = f_y = 4y^3$ $\qquad f_y(2,1) = 4(1)^3 = 4$

Normal line

$x = 2 - 12t$
$y = 1 - 4t$
$z = 13 + t$

Tangent plane

$-12(x-2) - 4(y-1) + 1(z-13) = 0$

$-12x - 4y + z = -15$

Example: Find an equation of the tangent plane $f(x, y) = \ln(5x + 2y)$ at the point $(-1, 3, 0)$. Also find a formula for the normal line at this point.

$$f_x = \frac{5}{5x+2y} \qquad f_x(-1,3) = \frac{5}{5(-1)+2(3)} = \frac{5}{-5+6} = \frac{5}{1} = 5$$

$\langle f_x, f_y, -1 \rangle$

$$f_y = \frac{2}{5x+2y} \qquad f_y(-1,3) = \frac{2}{1} = 2$$

$n = \langle -f_x, -f_y, 1 \rangle$

$n = \langle -5, -2, 1 \rangle$

normal line

$x = -1 - 5t$

$y = 3 - 2t$

$z = 0 + t$

Tangent plane

$-5(x+1) - 2(y-3) + 1(z-0) = 0$

Differentials:

$\frac{dy}{dx} = f'(x)$

$\Delta y =$ exact change

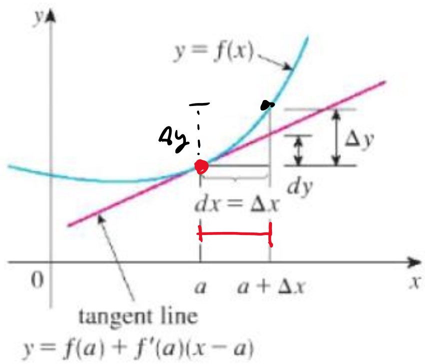
$dy =$ approx. change

In Cal I we had for $y = f(x)$ the differentials dy and dx defined as

$$dy = f'(x)dx$$

We saw that for the point (a, f(a)) and $dx = \Delta x$ we found that

$$f(a + \Delta x) = f(a) + \Delta y \approx \underbrace{f(a) + dy}_{f(a) + f'(a)\,dx}$$



Definition: Consider a function of two variables $z = f(x, y)$. Let Δx and Δy be increments of x and y, respectively.

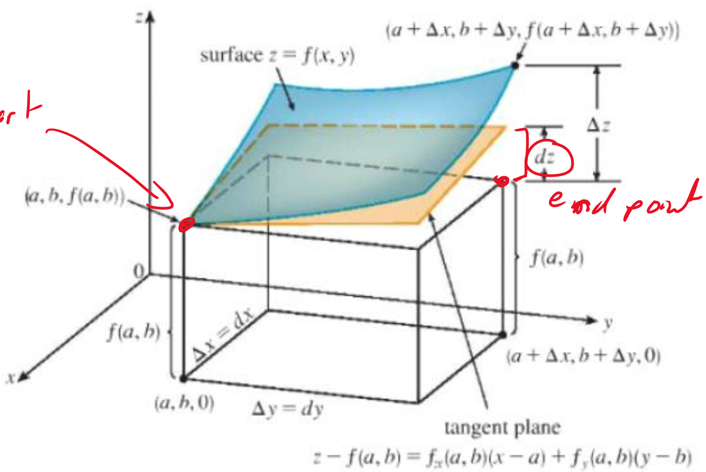
- Then **differentials dx and dy** are independent variables and $dx = \Delta x$ and $dy = \Delta y$
- The **differential dz**, also called the **total differential**, is a dependent variable and is defined by $dz = f_x(x, y)dx + f_y(x, y)dy$

Note: $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ and when Δx and Δy are small and the partials are both continuous, then $\Delta z \approx dz$.

Actual change

$\Delta z = f(\text{end pt.}) - f(\text{start pt}).$

Start



end point

$f(\text{endpt}) \approx f(\text{start pt}) + dz$

The linearization, $L(x, y)$, of the function $f(x, y)$ at the point (a, b) is

$$L(x, y) \approx \underbrace{f(a, b)}_{\text{start}} + \underbrace{f_x(a, b)(x - a) + f_y(a, b)(y - b)}_{dz}$$

Example: Use differentials to find an approximate value for $\sqrt{1.03^2 + 1.98^3}$

$f(x,y) = \sqrt{x^2+y^3}$

Start pt) $x = 1$, $y = 2$

$f(1,2) = \sqrt{1^2+2^3} = \sqrt{1+8} = \sqrt{9} = 3$

$f_x = \frac{1}{2}(x^2+y^3)^{-1/2} \cdot 2x$

$f_x = \frac{x}{\sqrt{x^2+y^3}}$

endpt) $x = 1.03$, $y = 1.98$

$dx = \Delta x = \overset{\text{end pt} - \text{start pt}}{1.03 - 1} = .03$

$dy = \Delta y = 1.98 - 2 = -.02$

$f_y = \frac{1}{2}(x^2+y^3)^{-1/2} \cdot 3y^2$

$f_y = \frac{3y^2}{2\sqrt{x^2+y^3}}$

$f_x(1,2) = \frac{1}{\sqrt{1^2+2^3}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

$f_y(1,2) = \frac{3(2)^2}{2\sqrt{1^2+2^3}} = \frac{12}{2\sqrt{9}} = \frac{6}{3} = 2$

Compute the total differential

$dz = f_x(1,2) \cdot dx + f_y(1,2)\,dy = \frac{1}{3}(.03) + 2(-.02)$

$= .01 - .04 = -.03$

$\sqrt{1.03^2+1.98^3} \approx f(1,2) + dz = 3 - .03 = \underline{\underline{2.97}}$

$L(x,y) \approx f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

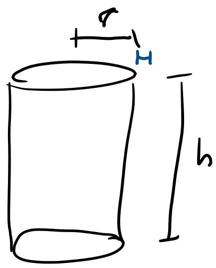
$= 3 + \frac{1}{3}(x-1) + 2(y-2)$

Example: Find the differential(i.e. the total differential) of the function
$w = x^5y^3 + x^2z^4$.

$$dw = w_x\,dx + w_y\,dy + w_z\,dz$$

$$dw = \left(5x^4y^3 + 2xz^4\right)dx + \left(x^5 \cdot 3y^2\right)dy + \left(x^2 \cdot 4z^3\right)dz$$

Example: Estimate the amount of material in a closed can (right circular cylinder) with a radius of 3 in and a height of 8 in if the material of the can is 0.04 in thick.



$V = \pi r^2 h$

$r = 3$ in

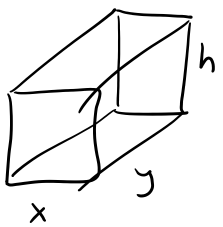
$h = 8$ in

$dr = .04$

$dh = \underbrace{.04}_{\text{Top}} + \underbrace{.04}_{\text{Bottom}} = .08$

$$\begin{aligned}
dV &= V_r\,dr + V_h\,dh \\
&= (2\pi r h)\,dr + (\pi r^2)\,dh \\
&= 2\pi(3)(8)(.04) + \pi(3)^2(.08) \\
&= 2.64\,\pi \\
&= 8.29 \text{ cubic in.}
\end{aligned}$$

Example: The dimensions of a closed rectangular box are measured as 90cm, 70cm, and 60cm with a possible error of 0.3cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.



$SA = 2xy + 2xh + 2yh$

$x = 90$
$y = 70$
$h = 60$

$dx = dy = dh = 0.3$

$dSA = \underbrace{SA_x}\,dx + \underbrace{SA_y}\,dy + \underbrace{SA_h}\,dh$

$= (2y + 2h)\,dx + (2x + 2h)\,dy + (2x + 2y)\,dh$

$= [2(70) + 2(60)](.3) + [2(90) + 2(60)](.3) + [2(90) + 2(70)](.3)$

$= 264$ sq cm.

Definition: A function $f(x, y)$ is **differentiable** at (a, b) if its partial derivatives f_x and f_y exist and are continuous at (a, b).

Note: Polynomial and rational functions are differentiable on their domains.

$f(x,y)$

normal vectors

$n = \langle -f_x, -f_y, 1\rangle$

$n = \langle f_x, f_y, -1\rangle$

Example: Find an equation of a tangent plane for the surface at the point $(1, 1, 1)$, if it is know the two space curves $\mathbf{r}(t)$ and $\mathbf{g}(s)$ are both both on the surface and they both go though the point $(1, 1, 1)$

$\hookrightarrow t=1 \qquad s=0$

$\mathbf{r}(t) = \langle t, t^2, t^3\rangle$

$\mathbf{g}(s) = \langle 1 + 2s, 1 + s - s^2, 1 - s + s^2 - s^3\rangle$

$1+2s=1$

$2s=0$

$s=0$

$r'(t) = \langle 1, 2t, 3t^2\rangle \qquad r'(1) = \langle 1, 2, 3\rangle$

$g'(s) = \langle 2, 1-2s, -1+2s-3s^2\rangle \qquad g'(0) = \langle 2, 1, -1\rangle$

$n = r'(1) \times g'(0) = \ldots = \langle -5, 7, -3\rangle$

tangent plane $\boxed{-5(x-1) + 7(y-1) - 3(z-1) = 0}$