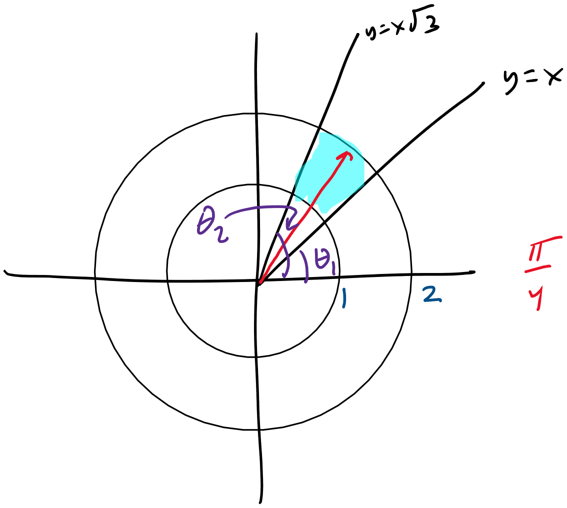
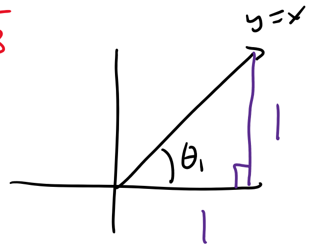
Example: Evaluate $\displaystyle\iint_D \arctan\left(\frac{y}{x}\right) dA$

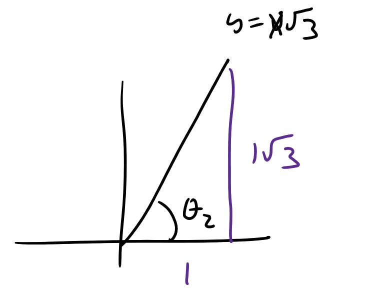
where $D = \{(x,y) \mid 1 \le x^2+y^2 \le 4,\ \underbrace{x \le y}_{y=x} \le \underbrace{x\sqrt{3}}_{y=x\sqrt{3}},\ x \ge 0\}$.



$1 \le r \le 2$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{3}$





$\tan\theta_1 = \frac{1}{1} = 1$

$\theta_1 = \frac{\pi}{4}$

$\tan\theta_2 = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\theta_2 = \frac{\pi}{3}$

$$f(x,y) = \arctan\left(\frac{y}{x}\right)$$
$$= \arctan\left(\frac{r\sin\theta}{r\cos\theta}\right)$$
$$= \arctan\left(\frac{\sin\theta}{\cos\theta}\right)$$
$$= \arctan(\tan\theta) = \theta$$

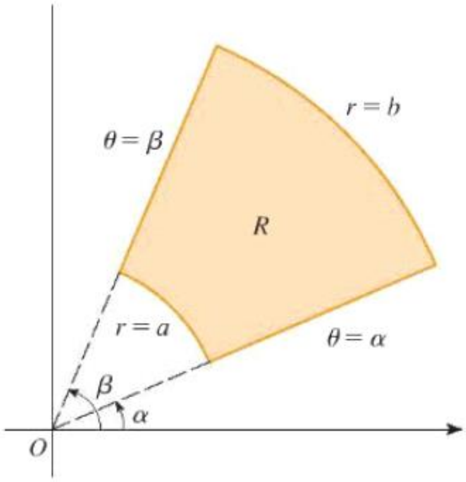
Region D is in 1st quadrant.

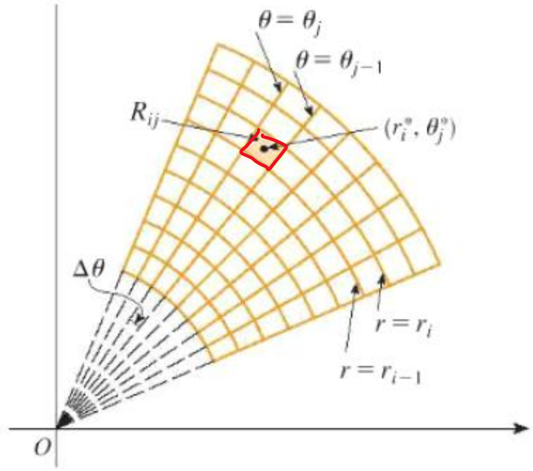
$$\iint_D \arctan\frac{y}{x}\, dA = \int_{\theta=\frac{\pi}{4}}^{\pi/3}\int_{r=1}^{2} \theta\, r\, dr\, d\theta = \int_{\theta=\frac{\pi}{4}}^{\pi/3} \theta\, d\theta \cdot \int_{r=1}^{2} r\, dr$$

$$= \frac{1}{2}\theta^2 \Big|_{\frac{\pi}{4}}^{\pi/3} \cdot \frac{1}{2}r^2 \Big|_1^2$$

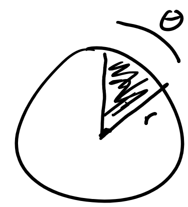
$$= \frac{1}{2}\left(\left(\frac{\pi}{3}\right)^2 - \left(\frac{\pi}{4}\right)^2\right) \cdot \frac{1}{2}(4-1) = \frac{7\pi^2}{192}$$

Pg 2: Polar integral derivation





$dA \quad \Delta A$



The center of the polar subrectangle $R_{ij} = \{(r,\theta) | r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$

has polar coordinates: $r_i^* = \frac{1}{2}(r_{i-1} + r_i)$ and $\theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$

Note: the area a sector of a circle with radius r and angle θ is $\frac{1}{2} r^2 \theta$.

$$\Delta A_{ij} = \frac{1}{2} r_i^2 \Delta\theta_j - \frac{1}{2} r_{i-1}^2 \Delta\theta_j = \frac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\Delta\theta_j$$

$$\Delta A_{ij} = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\,\Delta\theta_j = r_i^* \Delta r_i \Delta\theta_j$$

thus $dA = r\ dr d\theta$

Change to Polar Coordinates in a Double Integral: If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $c \le \theta \le d$, where $0 \le d - c \le 2\pi$, then

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(r\cos\theta, r\sin\theta)\ \ r\ drd\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

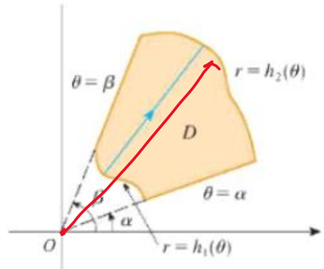
Pg 3: non-rectangular regions

If f is continuous on a polar region of the form

$D = \{(r,\theta) | \underbrace{c \le \theta \le d}, \underbrace{h_1(\theta) \le r \le h_2(\theta)}\}$

then

$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, \underline{rdrd\theta}$$



when to use polar

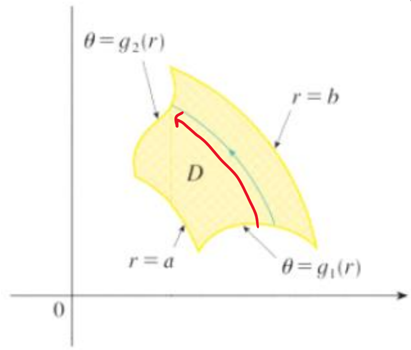
Region D is of a polar nature part of a circle

If the function contains x^2+y^2

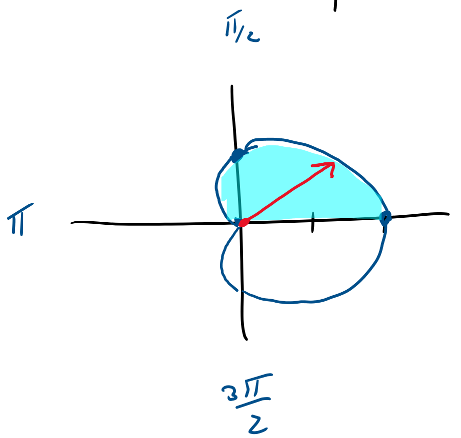
If $D = \{(r,\theta) | a \le r \le b, g_1(r) \le \theta \le g_2(r)\}$

then

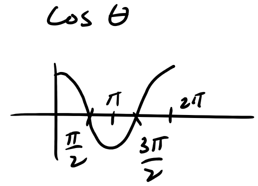
$$\iint_D f(x,y)dA = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)\, \underline{rd\theta dr}$$



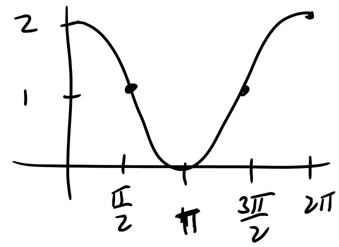
Example: Compute $\iint_D y\, dA$ where D is the upper half of the cardioid: $r = 1 + \cos\theta$.



$y = r\sin\theta$
$x = r\cos\theta$



$r = 1+\cos\theta$



$0 \le \theta \le \pi$

$0 \le r \le 1+\cos\theta$

$$\iint_D y\, dA = \int_{\theta=0}^{\pi}\int_{r=0}^{1+\cos\theta} r\sin\theta \cdot r\, dr\, d\theta = \int_{\theta=0}^{\pi}\int_{r=0}^{1+\cos\theta} r^2 \sin\theta\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \frac{1}{3} r^3 \sin\theta \Big|_{r=0}^{1+\cos\theta} d\theta = \int_{\theta=0}^{\pi} \frac{1}{3}(1+\cos\theta)^3 \sin\theta\, d\theta$$

$u = 1+\cos\theta$

$$= \dots = -\frac{1}{3}\frac{1}{4}(1+\cos\theta)^4 \Big|_{\theta=0}^{\pi}$$

$$= -\frac{1}{12}(1+\underbrace{\cos(\pi)}_{-1})^4 - -\frac{1}{12}(1+\underbrace{\cos(0)}_{1})^4 = -\frac{1}{12}(0)^4 + \frac{1}{12}(2)^4$$

$$= \frac{4}{3}$$

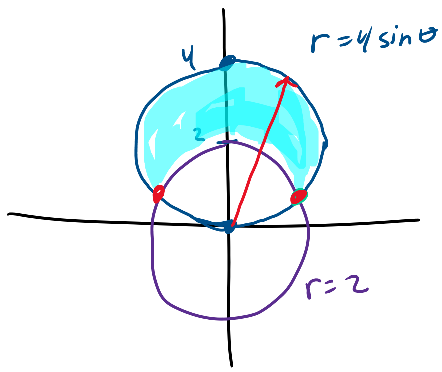
Pg 5

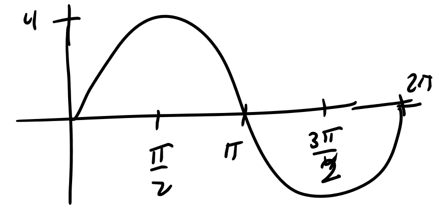
Example: Find the area of the region inside the circle $r = 4\sin\theta$ and out side the circle $r = 2$.

$r = 4\sin\theta$

$r^2 = 4r\sin\theta$

$x^2 + y^2 = 4y$





$2 \le r \le 4\sin\theta$

$\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$\text{Area} = \iint_D 1\, dA = \int_{\theta=\pi/6}^{5\pi/6} \int_{r=2}^{4\sin\theta} 1\, r\, dr\, d\theta$$

$$= \int_{\theta=\pi/6}^{5\pi/6} \frac{1}{2}r^2 \Big|_{r=2}^{4\sin\theta} d\theta = \int_{\theta=\pi/6}^{5\pi/6} \frac{1}{2}(4\sin\theta)^2 - \frac{1}{2}(2)^2\, d\theta$$

$$= \int_{\theta=\pi/6}^{5\pi/6} 8\sin^2\theta - 2\, d\theta = \int_{\theta=\pi/6}^{5\pi/6} 8\cdot\frac{1}{2}(1-\cos 2\theta) - 2\, d\theta$$

$$\boxed{\begin{aligned} \sin^2\theta &= \tfrac{1}{2}(1-\cos 2\theta) \\ \cos^2\theta &= \tfrac{1}{2}(1+\cos 2\theta) \end{aligned}}$$

$$= \ldots = \frac{4\pi}{3} + 2\sqrt{3}$$

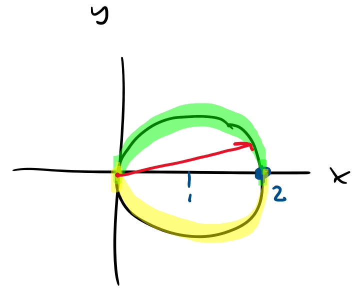
Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy−plane and inside the cylinder $x^2 + y^2 = 2x$.

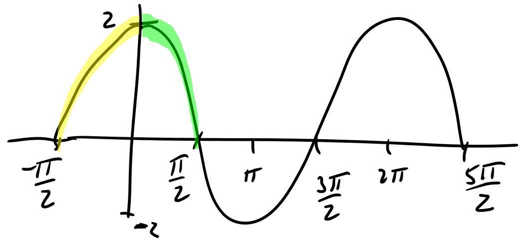
$$V = \iint_D x^2+y^2 \, dA$$

$$r^2 = 2r\cos\theta$$

$$r = 2\cos\theta$$

Region D.





$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$0 \le r \le 2\cos\theta$$

Could we use $\frac{3\pi}{2} \le \theta \le \frac{5\pi}{2}$? yes.

$$V = \iint_D x^2+y^2 \, dA = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2 \cdot r \, dr \, d\theta = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \int_{r=0}^{2\cos\theta} r^3 \, dr \, d\theta$$

$$= \dots = \frac{3\pi}{2}$$