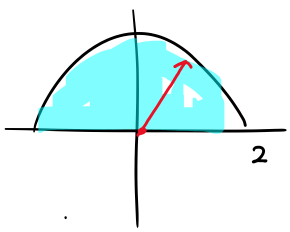
Section 15.4: Applications of Double Integrals

- **Area of a region D:** $\displaystyle\iint_D 1\,dA = A(D)$ the area of region D.
- **Volume of a solid** with base D and $f(x, y) \geq 0$ on region D: $\displaystyle\iint_D f(x, y)dA$
- The **total mass** of a lamina (thin plate or region) with variable density (in units of mass per unit area) on the region D is given by

  $$m = \iint_D \rho(x, y)dA$$

  where $\rho(x, y)$ is the density at the point (x, y).
  Note: If the density is constant then $m = \rho *$ Area of D.

Example: An electric charge is distributed over the part of the disk $x^2 + y^2 \leq 4$ in the top half of the xy–plane so that the charge density at (x, y) is $\sigma(x, y) = x^2 + y^2$ (measured in coulombs per square meter). Find the total charge on the disk.



$0 \leq r \leq 2$

$0 \leq \theta \leq \pi$

$$\text{Total charge} = \iint_D x^2+y^2 \, dA$$

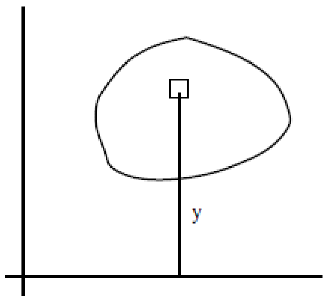
$$= \int_{\theta=0}^{\pi} \int_{r=0}^{2} r^2 \cdot r \, dr \, d\theta = \int_{\theta=0}^{\pi} \int_{r=0}^{2} r^3 \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi} 1 \, d\theta \cdot \int_{r=0}^{2} r^3 \, dr$$

$$= \pi \cdot \frac{1}{4} r^4 \Big|_0^2 = \ldots = 4\pi$$

Moments and Center of Mass

The moment of a particle about an axis is defined to be the mass of the particle times the distance of the particle from the axis.



$$M_x = \sum\sum \left(\rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij} \quad y_{ij}\right)$$

$$M_y = \sum\sum \left(\rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij} \quad x_{ij}\right)$$

Given a lamina with density function $\rho(x, y)$. The moment of the lamina about the $x-$axis, denoted M_x, and the moment about the $y-$axis, denoted M_y, is given by

$$M_x = \iint_D y\rho(x, y)dA \qquad\qquad M_y = \iint_D x\rho(x, y)dA$$

The **center of mass**, $(\overline{x}, \overline{y})$ of the lamina with density $\rho(x, y)$ are

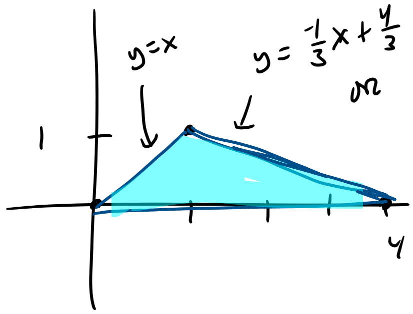
$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)dA \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)dA \qquad \text{where } m = \iint_D \rho(x, y)dA$$

Moments of Inertia: (second moment).

about the x-axis: $I_x = \iint_D y^2\rho(x, y)dA$ about the y-axis: $I_y = \iint_D x^2\rho(x, y)dA$

about the origin: $I_o = I_x + I_y = \iint_D (x^2 + y^2)\rho(x, y)dA$

Example: Find the mass of the lamina occupies the triangular region with vertices $(0,0)$, $(1,1)$, and $(4,0)$. The density of the region is given by $\underline{\rho(x,y) = y}$. Note the center of mass calculations can be found with the additional problems.



$$\text{mass} = m = \iint_D \rho(x,y)\,dA = \iint_D y\,dA$$

$$= \int_{y=0}^{1} \int_{x=y}^{4-3y} y\,dx\,dy = \cdots = \frac{2}{3}$$

$dy\,dx$? $\qquad \overrightarrow{dx\,dy}$

$0 \le y \le 1$

$y \le x \le 4-3y$

$$\bar{x} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA = \frac{1}{m}\int_{y=0}^{1}\int_{x=y}^{4-3y} x\cdot y\,dx\,dy = \cdots = \frac{3}{2}$$

$$\bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA = \frac{1}{m}\int_{y=0}^{1}\int_{x=y}^{4-3y} y\cdot y\,dx\,dy = \cdots = \frac{1}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{1}{2}\right)$$