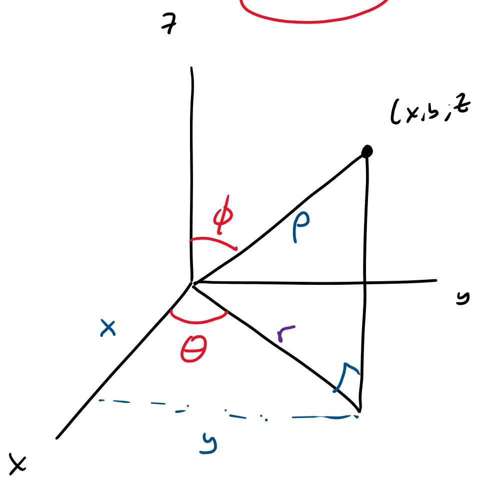
Section 15.8: Triple Integrals in Spherical Coordinates

Spherical Coordinates:
A Cartesian point (x, y, z) is represented by $\underbrace{(\rho, \theta, \phi)}$ in the Spherical Coordinate System where $\rho \geq 0$ and $0 \leq \phi \leq \pi$.



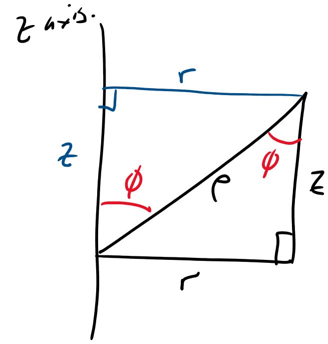
$\rho = \sqrt{x^2+y^2+z^2}$

$\rho^2 = x^2+y^2+z^2$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$



$\cos\phi = \dfrac{z}{\rho}$ $\quad\quad$ $\sin\phi = \dfrac{r}{\rho}$

$z = \rho\cos\phi$ $\quad\quad$ $r = \rho\sin\phi$

$$\boxed{\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2} \\ x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}}$$

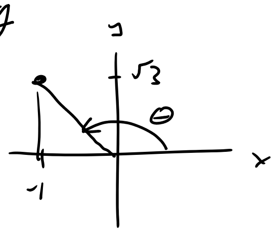
Example: Find the spherical coordinates for the points $(-1, \sqrt{3}, 2)$ and $(-1, \sqrt{3}, -2)$

$(-1, \sqrt{3}, 2)$

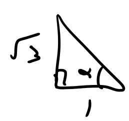
$$\rho = \sqrt{(-1)^2 + (\sqrt{3})^2 + (2)^2}$$
$$= \sqrt{1 + 3 + 4}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$

$$z = \rho \cos\phi$$
$$2 = \sqrt{8}\cos\phi$$
$$2 = 2\sqrt{2}\cos\phi$$
$$\cos\phi = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\phi = \frac{\pi}{4}$$

find θ



$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

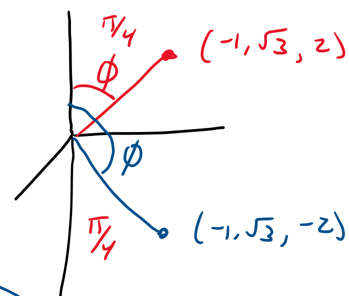


$$\tan\alpha = \frac{\sqrt{3}}{1}$$
$$\alpha = \frac{\pi}{3}$$

$$\begin{matrix} & \rho & \theta & \phi \end{matrix}$$
$$(-1, \sqrt{3}, 2) \Rightarrow \left(2\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4}\right)$$

$(-1, \sqrt{3}, -2)$

$$\rho = 2\sqrt{2}$$
$$\theta = \frac{2\pi}{3}$$
$$\phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$



$$(-1, \sqrt{3}, -2) \longrightarrow \left(2\sqrt{2}, \frac{2\pi}{3}, \frac{3\pi}{4}\right)$$

Example: Write the equations in spherical coordinates.

A) $x^2 + y^2 + z^2 = 25$ Sphere

$\rho^2 = 25$

$\rho = 5$

B) $z = 12 - 4x^2 - 4y^2$ paraboloid

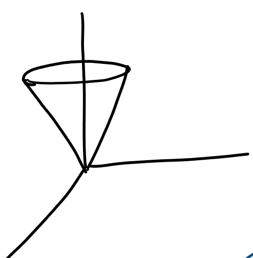
$$\rho\cos\phi = 12 - 4(\rho\sin\phi\cos\theta)^2 - 4(\rho\sin\phi\sin\theta)^2$$

$$= 12 - 4\rho^2\sin^2\phi\cos^2\theta - 4\rho^2\sin^2\phi\sin^2\theta$$

$$= 12 - 4\rho^2\sin^2\phi\{\cos^2\theta + \sin^2\theta\}$$

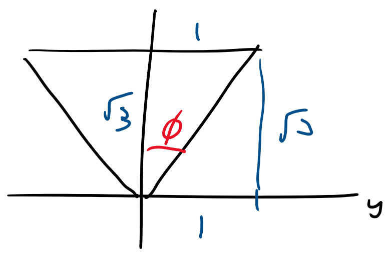
$$\rho\cos\phi = 12 - 4\rho^2\sin^2\phi$$

C) $z = \sqrt{3x^2 + 3y^2}$ Cone (top part)



$$\phi = \frac{\pi}{6}$$

method (2)



$z = \sqrt{3x^2 + 3y^2}$

Let $x = 0$

$z = \sqrt{3y^2}$

$z = \sqrt{3}\,y$

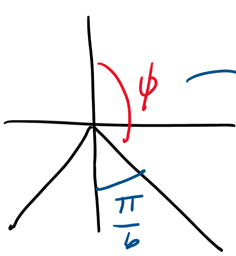
Let $y = 1$

$z = \sqrt{3}$

$\tan\phi = \frac{1}{\sqrt{3}}$

$\phi = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Cone $z = -\sqrt{3x^2 + 3y^2}$



$\phi \rightarrow \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Triple Integrals in Spherical Coordinates

In this coordinate system, the equivalent of a box is a spherical wedge
$E = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$

where $a \geq 0$, $\beta - \alpha \leq 2\pi$, and $d - c \leq \pi$

$$\iiint_E f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_a^b f(\overbrace{\rho \sin\phi \cos\theta}^{x}, \overbrace{\rho \sin\phi \sin\theta}^{y}, \overbrace{\rho \cos\phi}^{z})\, \rho^2 \sin\phi \; d\rho d\theta d\phi$$

Note: Spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region.

Example: Evaluate $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

Top $z=\sqrt{9-x^2-y^2}$ $\rightarrow$ $z^2=9-x^2-y^2$

$x^2+y^2+z^2=9$

top part of the sphere.

Bottom $z=0$

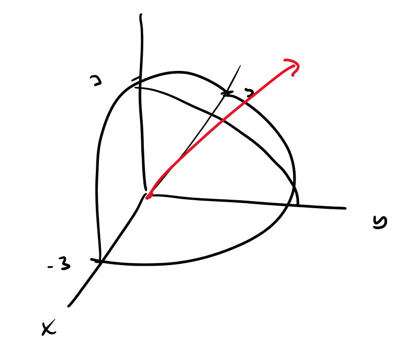
Region D

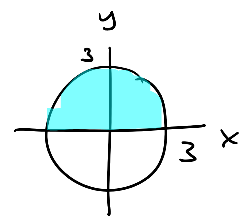
$0\le y\le\sqrt{9-x^2}$ $\quad -3\le x\le 3$

$y=\sqrt{9-x^2}$

$y^2=9-x^2$

$x^2+y^2=9$





$0\le\theta\le\pi$

$0\le\rho\le 3$

$0\le\phi\le\frac{\pi}{2}$

Example: Evaluate $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx = \int_{\theta=0}^{\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{3}(\rho\cos\phi)^2\sqrt{\rho^2}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

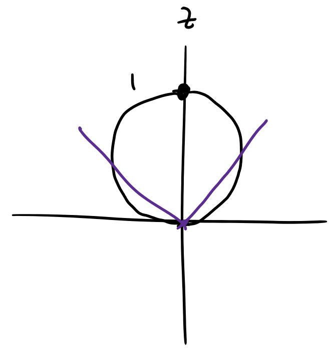
$$=\int_{\theta=0}^{\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{3}\rho^5\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\frac{81\pi}{2}$$

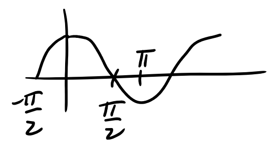
Example: Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $\underbrace{x^2 + y^2 + z^2 = z}$.

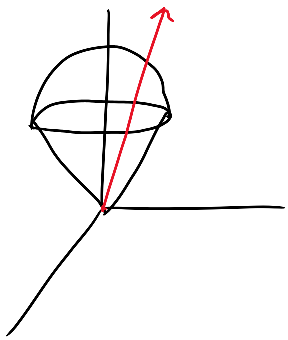
$\rho^2 = \rho \cos\phi$

$\rho = \cos\phi$



$\cos\phi$





$0 \le \theta \le 2\pi$

$0 \le \rho \le \cos\phi$

$0 \le \phi \le \frac{\pi}{4}$

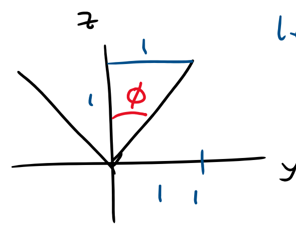
Cone

$z = \sqrt{x^2 + y^2}$

Let $x = 0$

$z = \sqrt{y^2}$

$z = y$



Let $y = 1$

$z = 1$

$\tan\phi = \frac{1}{1} = 1$

$\phi = \frac{\pi}{4}$

$V = \iiint_E 1 \, dV$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{\cos\phi} 1 \quad \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{\pi}{8}$$

Example: Convert the triple integral to spherical.

$$\int_{x=0}^{5}\int_{y=-\sqrt{25-x^2}}^{0}\int_{z=5-\sqrt{25-x^2-y^2}}^{5+\sqrt{25-x^2-y^2}} (x^2+y^2+z^2)^{1.5}\, dz\,dy\,dx$$

Top $z = 5 + \sqrt{25-x^2-y^2}$

Bottom $z = 5 - \sqrt{25-x^2-y^2}$

$\rightarrow$

$z - 5 = \sqrt{25-x^2-y^2}$

$(z-5)^2 = 25 - x^2 - y^2$

$x^2+y^2+(z-5)^2 = 25$

sphere of Radius 5 centered at $(0,0,5)$

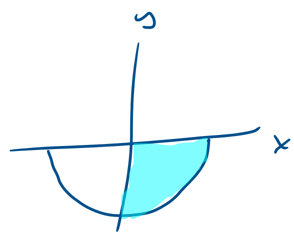
Region D

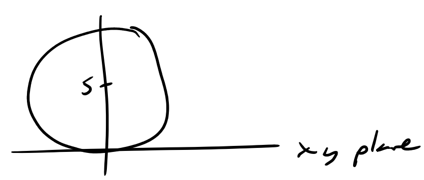
$0 \le y \le -\sqrt{25-x^2}$ $\quad$ $0 \le x \le 5$

$y = -\sqrt{25-x^2}$

$y^2 = 25 - x^2$

$x^2+y^2 = 25$





Solid is part of the sphere above Region D.

Sphere in spherical

$x^2+y^2+(z-5)^2 = 25$

$x^2+y^2+z^2-10z+25 = 25$

$x^2+y^2+z^2 = 10z$

$\rho^2 = 10\rho\cos\phi$

$\rho = 10\cos\phi$

$\frac{3\pi}{2} \le \theta \le 2\pi$ or $-\frac{\pi}{2} \le \theta \le 0$

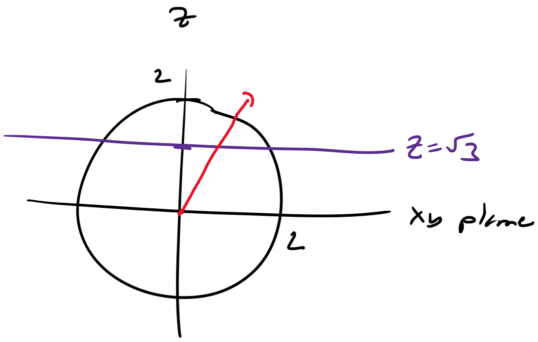
$0 \le \rho \le 10\cos\phi$

$0 \le \phi \le \frac{\pi}{2}$

Example: Convert the triple integral to spherical.

$$\int_{x=0}^{5}\int_{y=-\sqrt{25-x^2}}^{0}\int_{z=5-\sqrt{25-x^2-y^2}}^{5+\sqrt{25-x^2-y^2}} (x^2+y^2+z^2)^{1.5}\, dz\,dy\,dx = \int_{\theta=-\frac{\pi}{2}}^{0}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{10\cos\phi} \rho^{1.5}\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

Example Find the volume that is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the plane $z = \sqrt{3}$.



$0 \le \theta \le 2\pi$

$\sqrt{3}\sec\phi \le \rho \le 2$

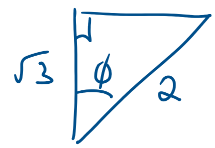
$0 \le \phi \le \frac{\pi}{6}$

$z = \sqrt{3}$

$\rho\cos\phi = \sqrt{3}$

$\rho = \frac{\sqrt{3}}{\cos\phi}$

$\rho = \sqrt{3}\sec\phi$



$\cos\phi = \frac{\sqrt{3}}{2}$

$\phi = \frac{\pi}{6}$

$$V = \iiint_E 1\, dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/6} \int_{\rho=\sqrt{3}\sec\phi}^{2} 1\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi\left(\frac{8}{3} - \frac{3}{2}\sqrt{3}\right)$$