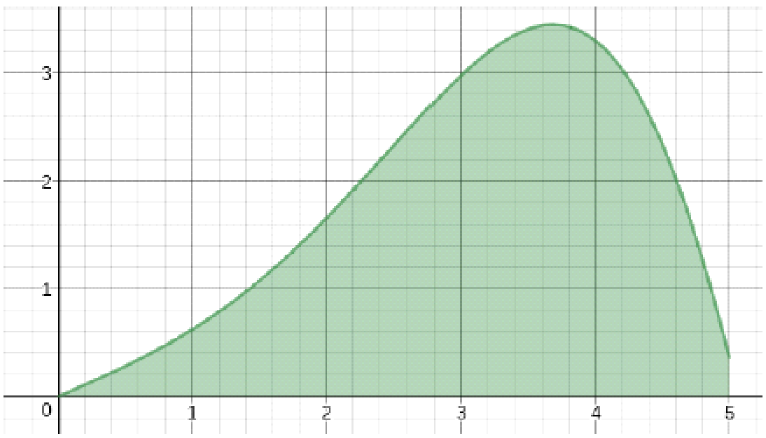
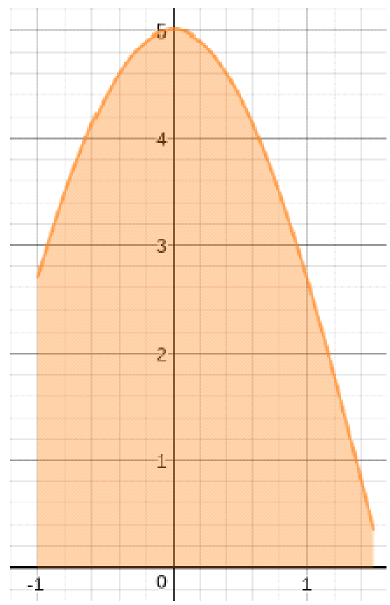
Section 15.9: Change of Variables in Multiple Integrals

From Cal 1/Cal 2 we know the following integrals are equivalent with the substitution $\underbrace{u = 0.1x^2 - 1}$.

$$\int_0^5 x\cos(0.1x^2 - 1)dx = \underbrace{\int_{-1}^{1.5} 5\cos(u)du}\approx 9.19483$$





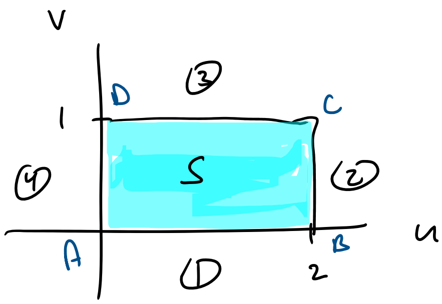
Consider $\iint\limits_R F(x,y)dA$ where R is a region in the $xy-$plane. Suppose we make the substitution $x = x(u,v)$ and $y = y(u,v)$ where x and y are functions of u and v that have continuous first-order partial derivatives. These equations give a **transformation** that will take a region S in the $uv-$plane and map it into a region R in the $xy-$plane, also called the image of S.

This will give $\displaystyle\iint\limits_R F(x,y)dA = \iint\limits_S F(x(u,v), y(u,v))dA$

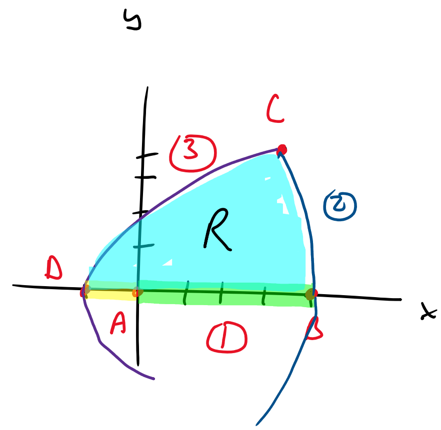
In order to find the region S that transforms into region R, we need the transformation be one-to-one. (This means no two points (u_1, v_1) and (u_2, v_2) map to the same point (x_1, y_1). Also needed is that as the boundary of S is traversed once, then the boundary of R will also be traversed only once.

Example: A transformation is defined by the equations $x = u^2 - v^2$, $y = 2uv$.

Find the image of the square $S = \{(u, v) | 0 \le u \le 2,\ 0 \le v \le 1\}$



(u,v)	(x,y)
A $(0,0)$	$(0,0)$
B $(2,0)$	$(4,0)$
C $(2,1)$	$(3,4)$
D $(0,1)$	$(-1,0)$



$\underbrace{x = u^2 - v^2}$ $\underbrace{y = 2uv}$

Side 1

$V = 0$ $\quad \underbrace{0 \le u \le 2}$

$\downarrow$

$y = 0$ $\quad \underbrace{x = u^2}$

$0 \le x \le 4$

Side 2

$u = 2 \quad 0 \le v \le 1$

$x = (2)^2 - v^2 \qquad y = 2(2)v$

$x = 4 - v^2 \qquad y = 4v$

$\underbrace{\frac{y}{4} = v}$

$x = 4 - \left(\frac{y}{4}\right)^2$

$x = 4 - \frac{1}{16}y^2$

Side 3

$V = 1 \quad 0 \le u \le 2$

$x = u^2 - (1)^2 \qquad y = 2u(1)$

$x = u^2 - 1 \qquad y = 2u$

$\underbrace{\frac{y}{2} = u}$

$x = \frac{1}{4}y^2 - 1$

Side 4

$u = 0 \quad 0 \le V \le 1$

$x = u^2 - v^2 \qquad y = 2uv$

$\underbrace{x = -v^2} \qquad y = 0$

$-1 \le x \le 0$

Pg 4: The Jacobian

Definition: The **Jacobian** of the transformation T given by $x = x(u,v)$ and $y = y(u,v)$ is

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \qquad \text{or}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

Example: Find the Jacobian of the transformation defined by $x = u^2 - v^2$, $y = 2uv$.

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 2u(2u) - (2v)(-2v)$$

$$= 4u^2 + 4v^2$$

Example: Find the Jacobian of the transformation defined by $x = r\cos\theta$, $y = r\sin\theta$.

$$J = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - -r\sin^2\theta$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

Change of Variable in a Double Integral Suppose T is a one-to-one transformation, where the substitutions have continuous first-order partial derivatives, whose Jacobian is nonzero and that maps a region S in the $uv-$plane onto a region R in the $xy-$plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Then

$$\iint\limits_R f(x,y)dA = \iint\limits_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

Abs. value of the the Jacobian.

Example: Let R be the region bounded by the $x-$axis and the parabolas $y^2 = 64 - 16x$ and $y^2 = 4 + 4x$, $y \geq 0$. Use the change of variables $x = u^2 - v^2$ and $y = 2uv$ to evaluate

$$\iint_R y \, dA$$

$-4 + y^2 = 4x$

$-1 + \frac{1}{4}y^2 = x$

$$J = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 - -4v^2$$

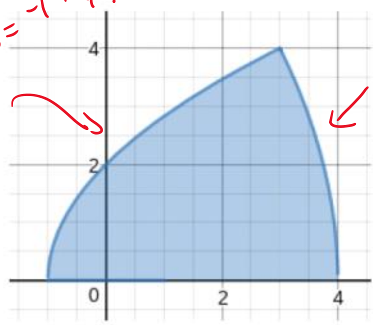
$$= 4u^2 + 4v^2$$

need $|J| = |4u^2 + 4v^2|$

$= 4u^2 + 4v^2$

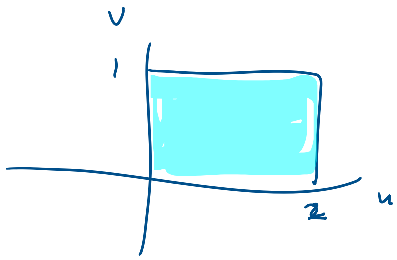
$x = -1 + \frac{1}{4}y^2$

$y^2 = 6Y - 16x$



Region R

Region S



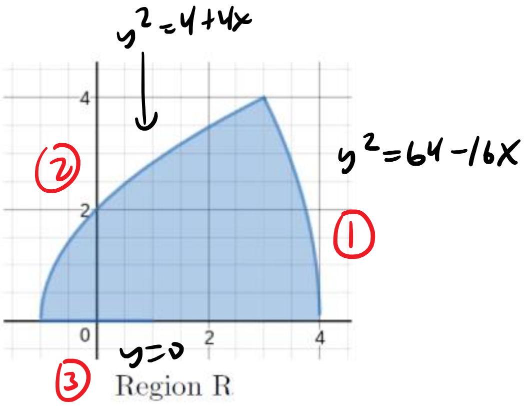
$0 \leq u \leq 2$

$0 \leq v \leq 1$

$$\iint_R y \, dA = \iint_S 2uv \cdot \underbrace{(4u^2 + 4v^2)}_{} \, dA = \int_{u=0}^{2} \int_{v=0}^{1} 8u^3 v + 8uv^3 \, dv \, du$$

$|J|$

Pg 8: part b



$$x = u^2 - v^2$$
$$y = 2uv$$

side 1

$$y^2 = 64 - 16x$$
$$(2uv)^2 = 64 - 16(u^2 - v^2)$$
$$4u^2v^2 = 64 - 16u^2 + 16v^2$$
$$u^2v^2 = 16 - 4u^2 + 4v^2$$
$$u^2v^2 + 4u^2 = 16 + 4v^2$$
$$u^2(v^2+4) = 4(4+v^2)$$
$$u^2(v^2+4) - 4(v^2+4) = 0$$
$$(u^2-4)(v^2+4) = 0$$

$u^2 - 4 = 0$ $\quad$ $v^2+4=0$

$u^2 = 4$ $\quad$ not possible.

$u = \pm 2$

side 2

$$y^2 = 4 + 4x$$
$$(2uv)^2 = 4 + 4(u^2 - v^2)$$
$$4u^2v^2 = 4 + 4u^2 - 4v^2$$
$$u^2v^2 = 1 + u^2 - v^2$$
$$u^2v^2 + v^2 = 1 + u^2$$
$$v^2(u^2+1) = u^2+1$$
$$v^2(u^2+1) - (u^2+1) = 0$$
$$(v^2-1)(u^2+1) = 0$$

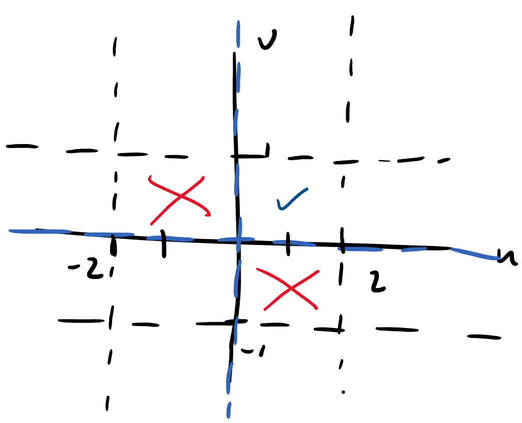
$v = \pm 1$ $\quad$ $u^2+1=0$ not possible

side 3

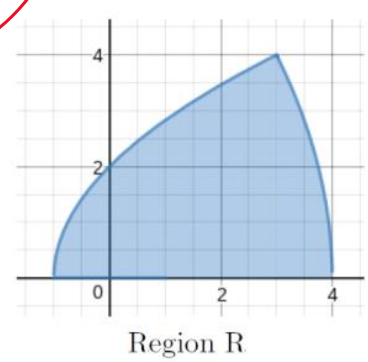
$$y = 0$$
$$2uv = 0$$
$$uv = 0$$

$u = 0$ OR $v = 0$

Region S



$$x = u^2 - v^2$$
$$y = 2uv$$
$$y \geq 0$$

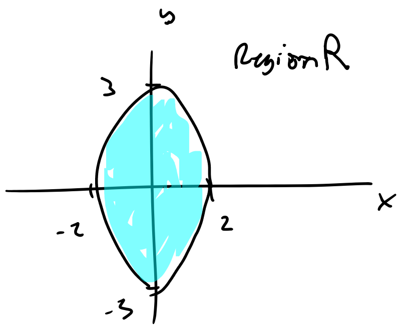


Example: Let R be the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Evaluate by making the given transformation.

$\iint\limits_R y + 3 \, dA$ with $\underbrace{x = 2u \text{ and } y = 3v}$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 - 0 = 6$$

$$|J| = |6| = 6$$

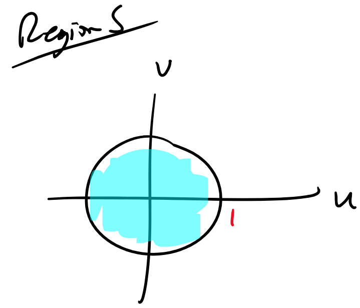


$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\frac{(2u)^2}{4} + \frac{(3v)^2}{9} = 1$$

$$\frac{4u^2}{4} + \frac{9v^2}{9} = 1$$

$$u^2 + v^2 = 1$$



Can use polar

$u = r\cos\theta$

$v = r\sin\theta$

$u^2 + v^2 = r^2$

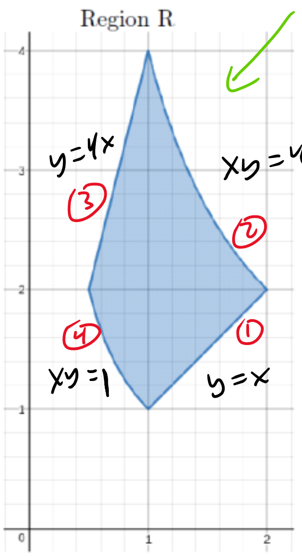
$$\iint\limits_R y + 3 \, dA = \iint\limits_S (3v+3) \cdot \underset{|J|}{6} \, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (3r\sin\theta + 3)\, 6\, r \, dr \, d\theta$$

$$= \dots = 18\pi$$

Example: Let R be the region in the first quadrant bounded by the lines $y = x$ and $y = 4x$ and the hyperbolas $xy = 1$ and $xy = 4$. Evaluate by making the given transformation.

$$\iint_R xy\, dA \text{ with } x = u/v \text{ and } y = v$$

all y values are > 0



$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{v}\cdot 1 - \frac{-u}{v^2}(0)$$

$$J = \frac{1}{v}$$

need $|J| = \frac{1}{|v|} = \frac{1}{v}$

$x = \frac{u}{v}$

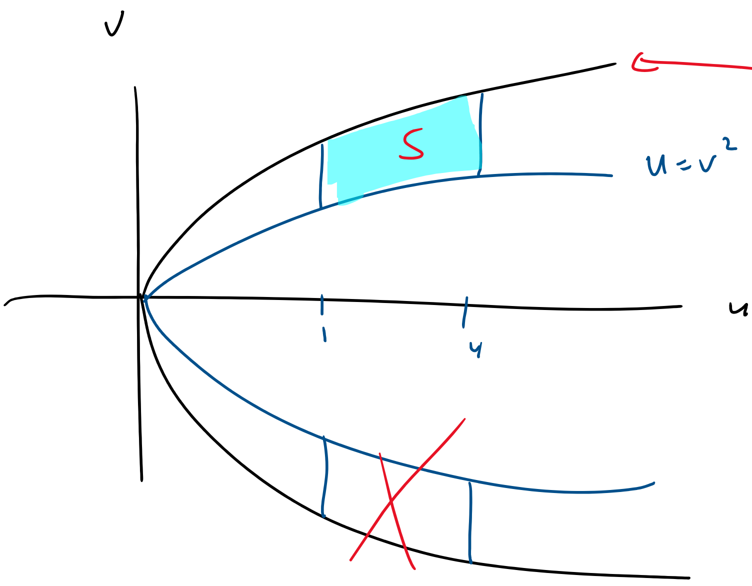
$y = v$

(1) $y = x$: $v = \frac{u}{v}$, $v^2 = u$

(2) $xy = 4$: $\frac{u}{v}\cdot v = 4$, $u = 4$

(3) $y = 4x$: $v = 4\frac{u}{v}$, $v^2 = 4u$, $u = \frac{1}{4}v^2$

(4) $xy = 1$: $\frac{u}{v}\cdot v = 1$, $u = 1$



$$\iint_R xy\, dA = \iint_S \frac{u}{v}\cdot v \cdot \overset{|J|}{\frac{1}{v}}\, dA = \int_{u=1}^{4}\int_{v=\sqrt{u}}^{\sqrt{4u}} \frac{u}{v}\, dv\, du$$

$$= \dots = \frac{15}{2} \ln(2)$$

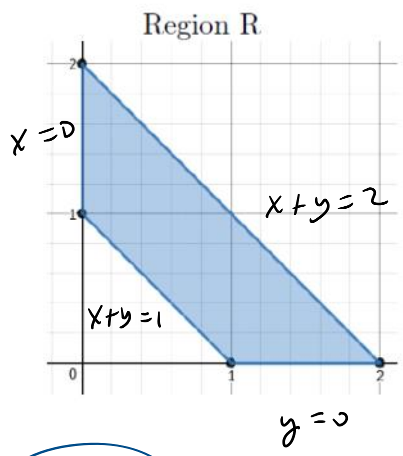
Example: Let R be the region in the xy–plane bounded by the vertices $(0,1)$, $(0,2)$, $(2,0)$, and $(1,0)$. Evaluate

$$\iint_R e^{(y-x)/(y+x)}\,dA$$

$e^{(y-x)/(y+x)} \to e^{u/v}$

$u = y-x$
$v = y+x$ → solve for $x + y$

Region R



$u = y-x$
$v = y+v$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4}$$

$J = -\frac{1}{2}$

$|J| = \left|-\frac{1}{2}\right| = \frac{1}{2}$

$v - u = (y+x) - (y-x)$
$= y + x - y + x$
$v - u = 2x$
$\boxed{x = \frac{1}{2}v - \frac{1}{2}u}$

$u + v = (y-x) + (y+x) = 2y$
$u + v = 2y$
$\boxed{y = \frac{1}{2}u + \frac{1}{2}v}$

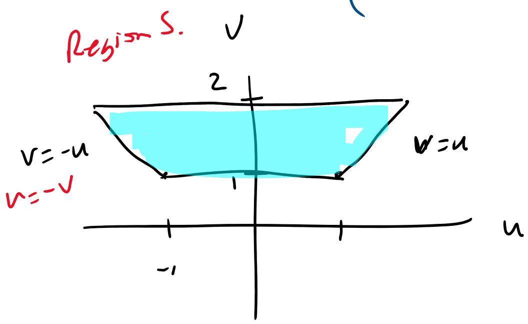
$x + y = 1$
$v = 1$

$x + y = 2$
$v = 2$

$y = 0$
$\frac{1}{2}u + \frac{1}{2}v = 0$
$\frac{1}{2}u = -\frac{1}{2}v$
$u = -v$
$v = -u$

$x = 0$
$0 = \frac{1}{2}v - \frac{1}{2}u$
$\frac{1}{2}u = \frac{1}{2}v$
$u = v$
$v = u$

Region S.



$du\,dv$
$1 \le v \le 2$
$-v \le u \le v$

Example: Let R be the region in the xy–plane bounded by the vertices $(0,1)$, $(0,2)$, $(2,0)$, and $(1,0)$. Evaluate

$$\iint_R e^{(y-x)/(y+x)}\,dA = \int_{v=1}^{2}\int_{u=-v}^{v} e^{u/v}\cdot\frac{1}{2}\,du\,dv = \cdots = \frac{3}{4}\left(e^{1} - e^{-1}\right)$$

$v - 1$ $u = -V$

$\uparrow$

$|J|$

Triple Integrals

Given the transformation $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$ then the Jacobian is the following 3×3 determinant.

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

With a hypotheses similar to the double integral change of variables we have the following for the triple integral.

$$\iiint\limits_R f(x, y)dV = \iiint\limits_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw$$

Example: Find the Jacobian for the transformation.

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

We compute the Jacobian as follows:

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \end{vmatrix} - \rho \sin\phi \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta \end{vmatrix}$$

$$= \cos\phi \, (-\rho^2 \sin\phi \cos\phi \sin^2\theta - \rho^2 \sin\phi \cos\phi \cos^2\theta)$$

$$- \rho \sin\phi \, (\rho \sin^2\phi \cos^2\theta + \rho \sin^2\phi \sin^2\theta)$$

$$= -\rho^2 \sin\phi \cos^2\phi - \rho^2 \sin\phi \sin^2\phi = \underbrace{-\rho^2 \sin\phi}$$

Since $0 \leqslant \phi \leqslant \pi$, we have $\sin\phi \geqslant 0$. Therefore

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = \underbrace{\left| -\rho^2 \sin\phi \right|} = \underbrace{\rho^2 \sin\phi}$$