Section 16.5: Curl and Divergence

Definition: The del operator, denoted ∇, is defined as $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

Definition: If $\mathbf{F} = \langle P, Q, R \rangle$ is a a vector field on $\Re^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of $\mathbf{F}$ is the vector field on $\Re^3$ defined by

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$$\text{curl } \mathbf{F} = \langle R_y - Q_z, \ \overset{-(R_x - P_z)}{P_z - R_x}, \ Q_x - P_y \rangle$$

$$F = \langle \overset{x}{P}, \overset{y}{Q}, \overset{z}{R} \rangle$$

Note: The curl $\mathbf{F}$ relates to rotation of a fluid at point P around the axis that points in the same direction of curl $\mathbf{F}$.

Note: If the curl $\mathbf{F} = \mathbf{0}$ at point P, then $\mathbf{F}$ is called **irrotational** at P.

$\quad\quad\quad\quad P \quad\quad Q \quad\quad R$

Example: Find the curl of the vector field $\mathbf{F} = \langle x^2 y, yz^2, zx^2 \rangle$

$$\text{curl } \mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

$$\text{curl } \mathbf{F} = \langle 0 - 2yz,\ 0 - 2zx,\ 0 - x^2 \rangle$$

$$= \langle -2yz,\ -2zx,\ -x^2 \rangle$$

Theorem: If f is a function of three variables that has continuous second-order partial derivatives, then $\text{curl}(\nabla f) = \mathbf{0}$.

$P = f_x$ $\qquad$ $P_y = f_{xy}$

$$\nabla f = \langle P, Q, R \rangle = \langle f_x, f_y, f_z \rangle$$

$$\text{curl}(\nabla f) = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

$$\text{curl}(\nabla f) = \langle f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy} \rangle = \langle 0,\ 0,\ 0 \rangle$$

Theorem: If $\mathbf{F}$ is a vector field defined on all of $\Re^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

Example: Determine if the vector field is conservative. $\mathbf{F} = \langle \overset{P}{zx}, \overset{Q}{xy}, \overset{R}{yz} \rangle$.

$$\text{curl } \mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

$$\text{curl } F = \langle z - 0,\ x - 0,\ y - 0 \rangle$$

$$= \langle z, x, y \rangle$$

not conservative

Pg 4: divergence

Definition: If $\mathbf{F} = \langle P, Q, R \rangle$ is a a vector field on $\Re^3$ and P_x, Q_y and R_z exist, then the **divergence of F** is the scalar function given by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$$

$\text{div } F = P_x + Q_y + R_z$

Note: If $\mathbf{F}$ is a velocity field for a fluid, then div $\mathbf{F}$ at a point measures the tendency of the fluid to diverge from that point. div $\mathbf{F}$ positive(negative) means move away(towards).

Example: Compute the div $\mathbf{F}$ where $\mathbf{F} = \langle \overset{P}{y^2z^3}, \overset{Q}{2xyz^3}, \overset{R}{3xy^2z^2} \rangle$.

$\text{div } F = P_x + Q_y + R_z = 0 + 2xz^3 + 6xy^2z$

Theorem: If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on $\Re^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div}\left(\text{curl } \mathbf{F}\right) = \text{div}\, \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

$$\text{div}\left(\text{curl } \mathbf{F}\right) = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

$$\text{div}\left(\text{curl } \mathbf{F}\right) = \underline{0}$$

Example: Is there a vector field $\mathbf{G}$ on $\Re^3$ such that curl $\mathbf{G} = \underline{\langle yz, xyz, zy \rangle}$? $\underline{\text{No!}}$

$$\text{div} \langle \overset{P}{yz}, \overset{Q}{xyz}, \overset{R}{yz} \rangle = \overset{P_x}{0} + \overset{Q_y}{xz} + \overset{R_z}{y}$$

Pg 6: interesting information

Example: Compute curl $\mathbf{F}$ if $\mathbf{F}$ is a vector field on $\Re^2$.

Suppose that $F = \langle P, Q \rangle$. Lets expand this to three dimensions by letting the z component, R, be zero.

i.e. $F = \langle P, Q, 0 \rangle$.

Thus curl$\mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle = \langle 0,\ 0,\ Q_x - P_y \rangle$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \iint_D Q_x - P_y dA = \iint_D \text{curl } \mathbf{F} \cdot \mathbf{k}\ dA$$