Section 12.4: The Cross Product

Reviewing the Determinate

The determinate of a 2x2 matrix is computed by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinate of a 3x3 matrix is computed by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example: Find the determinate of this matrice.

$$\begin{vmatrix} 1 & 3 & 4 \\ 5 & 0 & 2 \\ -3 & 6 & 7 \end{vmatrix} = 1 \begin{vmatrix} 0 & 2 \\ 6 & 7 \end{vmatrix} - 3 \begin{vmatrix} 5 & 2 \\ -3 & 7 \end{vmatrix} + 4 \begin{vmatrix} 5 & 0 \\ -3 & 6 \end{vmatrix}$$

$$= 1\left[0(7) - 2(6)\right] - 3\left[5(7) - 2(-3)\right] + 4\left[5(6) - 0(-3)\right]$$

$$= -12 - 3(35 + 6) + 4(30)$$

$$= -12 - 123 + 120$$

$$= -15$$

Example: Find the determinate of this matrice.

$$\left|\begin{array}{ccc} 1 & 3 & 4 \\ 5 & 0 & 2 \\ -3 & 6 & 7 \end{array}\right| \begin{array}{cc} 1 & 3 \\ 5 & 0 \\ -3 & 6 \end{array}$$

$$1(0)(7) + 3(2)(-3) + 4(5)(6) - (-3)(0)(4) - (6)(2)(1) - (7)(5)(3)$$

$$= 0 - 18 + 120 - 0 - 12 - 105$$

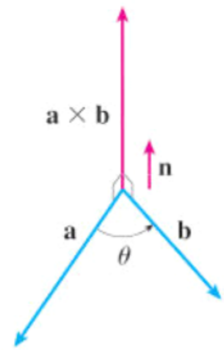
$$= -30 + 120 - 105 = -15$$

Pg 2: cross product

Definition: If $\mathbf{a}$ and $\mathbf{b}$ are two nonzero three-dimensional vectors, the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}| \sin(\theta))\, \mathbf{n}$$

where θ is the angle, $0 \leq \theta \leq \pi$, between $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{n}$ is a **unit vector** perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and whose direction is given by the **right-hand rule**: If the fingers of your right hand curls through the angle θ from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{n}$.



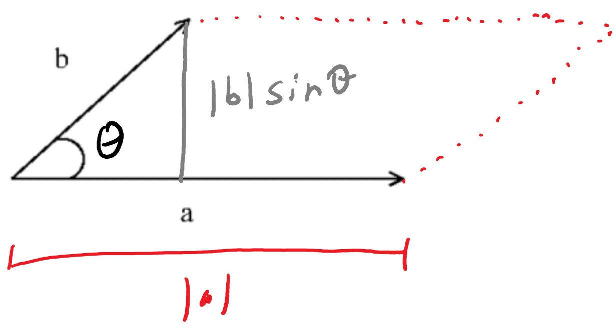
Note: $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$

Note: Two non-zero vectors,$\mathbf{a}$ and $\mathbf{b}$, are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

$a \times b = (|a||b| \sin\theta)n$

$|a \times b| =$ area of the parallelogram.

Geometric Interpretation:



Properties of the Cross Product: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and d is a scalar, then

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}) = -\mathbf{b} \times \mathbf{a}$
- $(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b})$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

Definition: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2,\ \ a_3 b_1 - a_1 b_3,\ \ a_1 b_2 - a_2 b_1 \rangle$$

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$a_2 b_3 - a_3 b_2$

Example: Compute the following for the vectors $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, -5, 6 \rangle$.

A) $\mathbf{a} \times \mathbf{b}$ $= \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & -5 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ -5 & 6 \end{vmatrix} i - \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} j + \begin{vmatrix} 1 & 3 \\ 2 & -5 \end{vmatrix} k$

$= (18 - -20) i - (6 - 8) j + (-5 - 6) k$

$= 38i + 2j - 11k = \langle 38, 2, -11 \rangle$

B) $\mathbf{b} \times \mathbf{a}$ $= -(a \times b) = \langle -38, -2, 11 \rangle$

C) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ $= 1(38) + 3(2) + 4(-11) = 38 + 6 - 44 = \underline{0}$

Example: Find a vector orthogonal to the plane determined by the points $A(1,2,3)$, $B(4,6,8)$, and $C(15,2,-5)$

$$\begin{array}{ccccc} i & j & k & i & j \\ 3 & 4 & 5 & 3 & 4 \\ 14 & 0 & -8 & 14 & 0 \end{array}$$

$$\vec{AB} = \langle 3, 4, 5 \rangle$$

$$\vec{AC} = \langle 14, 0, -8 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 3 & 4 & 5 \\ 14 & 0 & -8 \end{vmatrix}$$

$$= -32i + 70j + 0k - 56k - 0i - -24j$$

$$= \langle -32, 94, -56 \rangle$$

Example: Find the area of the parallelogram with vertices: $P(1, 1, 2)$, $Q(6, 1, 2)$, $R(4, 5, 5)$, and $S(9, 5, 5)$

$$a = \vec{PQ} = \langle 5, 0, 0 \rangle$$

$$b = \vec{PR} = \langle 3, 4, 3 \rangle$$

$$a \times b = \dots = \langle 0, -15, 20 \rangle$$

$$\text{Area} = |a \times b| = \sqrt{0 + 225 + 400} = \sqrt{625} = 25$$

Example: Find the area of the triangle determined by the points $P(1, 1, 2)$, $Q(6, 1, 2)$, and $R(4, 5, 5)$.

$a = \overrightarrow{PQ}$

$b = \overrightarrow{PR}$

$$\text{Area} = \frac{1}{2} |a \times b|$$

$$= \frac{1}{2} 25 = 12.5$$

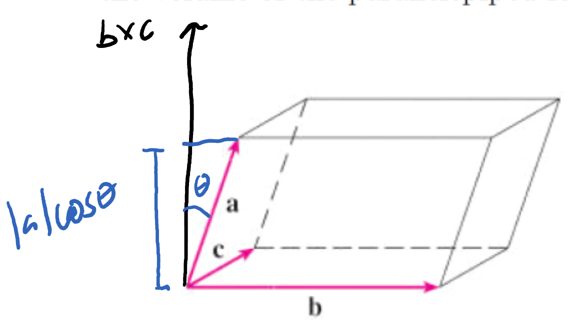
Definition: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ are vectors, then the **scalar triple product** is given by

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k} \right)$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note: The geometric interpretation of scalar triple product is that its magnitude is the volume of the parallelepiped formed by the vectors: **a**, **b**, and **c**.



$|b \times c|$ = Area of base.

$$a \cdot (b \times c) = |a|\,|b \times c| \cos\theta$$
$$= |b \times c|\,|a| \cos\theta$$

$|a \cdot (b \times c)|$ = Area of parallelepiped

"BOX"

$a \cdot (c \times b)$

$b \cdot (a \times c)$

Example: Compute a scalar triple product of these vectors: $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle 4, 5, 6 \rangle$, and $\mathbf{c} = \langle 2, 7, 5 \rangle$ Are these vectors co-planer?

$$a \cdot (b \times c) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 7 & 5 \end{vmatrix} = 25 + 24 + 84 - (30 + 42 + 40)$$

$$= 133 - 112$$

$$= 21$$

$$\begin{array}{r} {}^{2}28 \\ 3 \\ \hline 84 \end{array}$$

$\underline{30}$

Example: Determine if these points are co-planer: $A(4, -3, 1)$, $B(6, -4, 7)$, $C(1, 2, 2)$, and $D(0, 1, 11)$

$a = \overrightarrow{AB} = \langle 2, -1, 6 \rangle$

$b = \overrightarrow{AC} = \langle -3, 5, 1 \rangle$

$c = \overrightarrow{AD} = \langle -4, 4, 10 \rangle$

$a \cdot (b \times c) = \cdots = 114$

not coplaner.