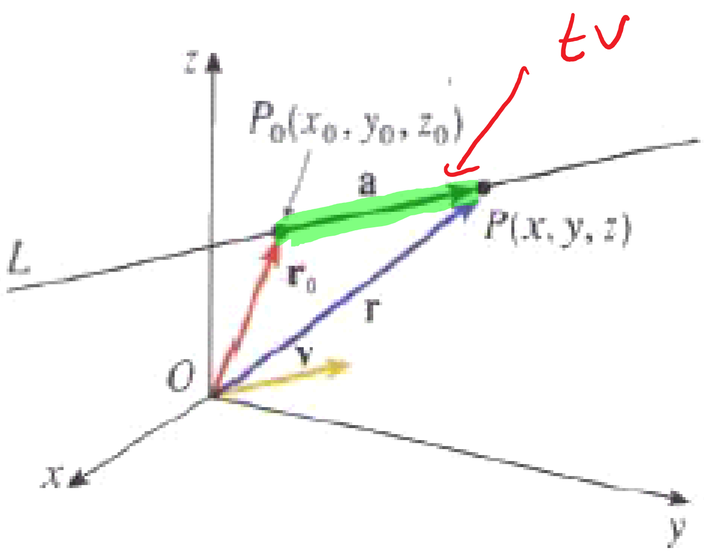
Section 12.5: Equations of Lines and Planes

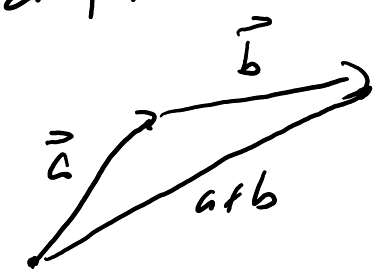
Definition: The vector equation of a line is found by the formula

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}_0$ is a vector representation of a point on the line, $\mathbf{v}$ is a directional vector of the line (i.e. a vector that is parallel to the line), and $t \in \Re$.



$\vec{a} + \vec{b}$



Example: Find the vector equation and the parametric equations of a line though the point $(1, 2, 3)$ where the line is parallel to the vector $\mathbf{v} =< 2, 5, 10 >$.

$r_0 + tv$

$r = \langle 1, 2, 3 \rangle + t \langle 2, 5, 10 \rangle$

$r = \langle 1+2t, 2+5t, 3+10t \rangle$ vector eq.

$x = 1+2t$

$y = 2+5t$

$z = 3+10t$

parametric equations.

A

Example: Find the vector equation of the line through the points $(3,5,5)$ and
B $(2,1,-5)$. Also give the parametric equations of this line. Where does the line intersect the xy-plane?

$r = r_0 + tv$

$v = \vec{BA}$

$v = \langle 1, 4, 10 \rangle$

$r = \langle 3,5,5 \rangle + t\langle 1,4,10 \rangle$

$r = \langle 3+t, 5+4t, 5+10t \rangle$ vector eq.

$$\left.\begin{aligned} x &= 3+t \\ y &= 5+4t \\ z &= 5+10t \end{aligned}\right\}$$ parametric equations

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$z = 0$

$5 + 10t = 0$

$10t = -5$

$t = -\frac{1}{2}$

$r(-\frac{1}{2}) = \langle 2.5, 3, 0 \rangle$

point. $(2.5, 3, 0)$

Example: Is the point $(7, 10, 17)$ on the line $\mathbf{r} = \langle 1+3t,\ 2+4t,\ 3+7t \rangle$?

$$1 + 3t = 7$$
$$3t = 6$$
$$t = 2$$

$$r(2) = \langle 7, 10, 17 \rangle$$

yes.

Pg 5: symmetric equations

Symmetric equations of a line: If a, b, c $\neq 0$ and line L goes through the point (x_0, y_0, z_0) with directional vector $\langle a, b, c\rangle$ then

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t$$

If, for example, $a = 0$ then the symmetric equations have the form:

$$x = x_0, \ \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$r = r_0 + tv$

Example: Find the symmetric equations of the line through the point $(5, 8, -2)$ and parallel to the line

$$\boxed{\begin{aligned} x &= 2+4t \\ y &= 3+2t \\ z &= 1+6t \end{aligned}} \longrightarrow v = \langle 4, 2, 6 \rangle$$

Answer: $\frac{x-5}{4} = \frac{y-8}{2} = \frac{z+2}{6}$

Definition: Skew lines are lines that are not parallel and do not intersect.

Example: Are these lines parallel, skew, or intersecting? If intersecting, find the point of intersection.

$$L_1: \frac{x+2}{3} = \frac{y-5}{-4} = \boxed{\frac{1-z}{2}} = m$$

and

$$L_2: \quad x = 1-t, \quad y = 3+2t, \quad z = -12-3t$$

$\frac{1-z}{2} = m$

$1-z = 2m$

$$\boxed{\begin{aligned} x &= -2+3m \\ y &= 5-4m \\ z &= 1-2m \end{aligned}}$$

$V_1 = \langle 3, -4, -2 \rangle \qquad V_2 = \langle -1, 2, -3 \rangle$

not parallel.

~~x~~ $1-t = -2+3m$

$3-t = 3m$

$3-3m = t$

$t = -3$

~~y~~ $3+2t = 5-4m$

$3+2(3-3m) = 5-4m$

$3+6-6m = 5-4m$

$9-5 = 2m$

$4 = 2m$

$m = 2$

$-12-3t = 1-2m$

$z = 1-2m$

$z = 1-4 = -3$

$z = -12-3t$

$z = -12+9 = -3$

Intersection point $(4, -3, -3)$

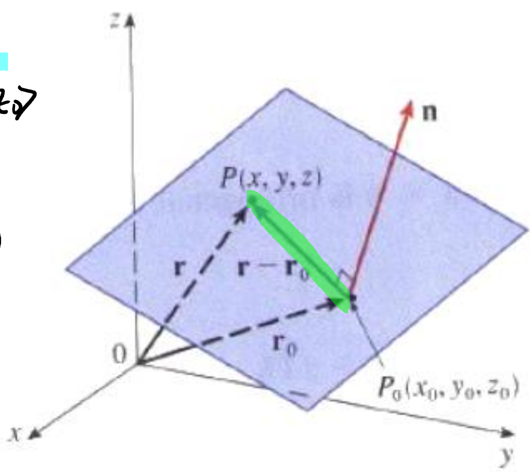
Pg 7: planes

A **plane** is determined by a point $P_0(x_0, y_0, z_0)$ and a vector $\mathbf{n} =< a, b, c >$ that is orthogonal to the plane. The vector $\mathbf{n}$ is called a normal vector.

$r = \langle x, y, z \rangle$

$r_0 = \langle x_0, y_0, z_0 \rangle$

$P(x, y, z)$



$n \cdot (r - r_0) = 0$

$n \cdot r - n \cdot r_0 = 0$

$n \cdot r = n \cdot r_0$

$$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{d}$$

$$ax + by + cz = d$$

Vector equation of the plane:

$n \cdot (r - r_0) = 0$ or $n \cdot r = n \cdot r_0$

Scalar equation of the plane:

Example: Find an equation of the plane through the point $(1, 2, 3)$ and is orthogonal to the vector $\langle 3, 4, 7 \rangle = n$

$$3(x-1) + 4(y-2) + 7(z-3) = 0$$

$$3x + 4y + 7z = 3(1) + 4(2) + 7(3)$$

$$3x + 4y + 7z = 32$$

Example: Find an equation of the plane through the points $A(1,1,3)$, $B(-1,3,2)$, and $C(1,-1,2)$.

$n = ?$ point in the plane ✓

$\overrightarrow{AB} = \langle -2, 2, -1 \rangle$

$\overrightarrow{AC} = \langle 0, -2, -1 \rangle$

$\overrightarrow{AB} \times \overrightarrow{AC} = \langle -4, -2, 4 \rangle = n$

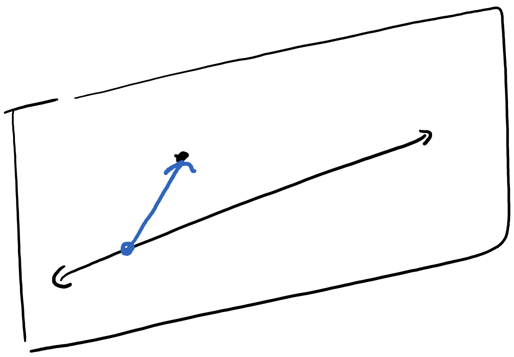
$-4x - 2y + 4z = -4(1) + -2(1) + 4(3)$

$$\boxed{-4x - 2y + 4z = 6}$$

Example Find an equation of the plane through the point $\overset{A}{(1,2,3)}$ and contains the line $x = 2 + 4t$, $y = 1 + 5t$, $z = -1 + 3t$

$n = ?$

point $= ?$ ✓



$v = \langle 4, 5, 3 \rangle$

let $t = 0$ point $\overset{B}{(2, 1, -1)}$

$\vec{AB} = \langle 1, -1, -4 \rangle$

$n = \vec{AB} \times v = \langle 17, -19, 9 \rangle$

$$17x - 19y + 9z = 6$$

Example: You are given two lines. Does there exist a plane that contains the given lines? If not, what conditions are needed so that there is a plane that contains the given lines?

no. Skew lines can not be in the same plane.

yes. if Intersecting lines

or parallel lines

Definition: Two planes are parallel if their normal vectors are parallel.

Definition: Two planes are perpendicular(orthogonal) if their normal vectors are perpendicular.

Definition: The angle between two non-parallel planes is the acute angle between the normal vectors.

Example: Determine if the pairs of planes are parallel, orthogonal, or neither?

P_1 : $4x + 2y - 8z = 15$ $\qquad n_1 = \langle 4, 2, -8 \rangle$

P_2 : $2x + y - 4z = 12$ $\qquad n_2 = \langle 2, 1, -4 \rangle$

P_3 : $3x + 2y + 2z = 10$ $\qquad n_3 = \langle 3, 2, 2 \rangle$

n_1 + n_2 are parallel

P_1 + P_2 parallel

$n_1 \cdot n_3 = 12 + 4 - 16 = 0$

P_1 + P_3 are perp.

P_2 + P_3 are perp.

Example: Find an equation of the line of intersection, L, of these two planes. $r = \underline{\underline{r_0}} + t\,\underline{\underline{v}}$

P_1: $x - y + 3z = 0$

P_2: $x + y + 4z = 2$

Let $\underline{\underline{z=0}}$ $\quad x - y = 0 \rightarrow x = y$

$x + y = 2 \rightarrow 2x = 2$

$x = 1 \rightarrow y = 1$

$\underline{(1, 1, 0) \text{ point}}$

$n_1 = \langle 1, -1, 3 \rangle$ $\qquad n_2 = \langle 1, 1, 4 \rangle$

v is in both planes.

$n_1 \times n_2 = \langle -7, -1, 2 \rangle = v$

$$\boxed{L = \langle 1 - 7t, 1 - t, 2t \rangle}$$

Pg 15: distance

The distance between a point $P\ (x, y, z)$ to the plane $ax + by + cz + d = 0$ is

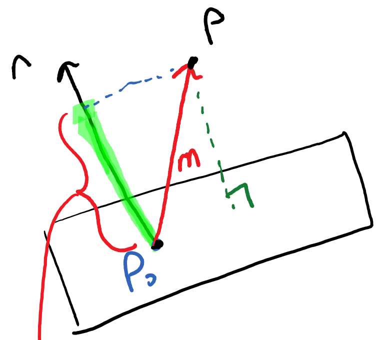
$P_0 (x_0, y_0, z_0)$

point on the plane

$ax_0 + by_0 + cz_0 + d = 0$

$ax_0 + by_0 + cz_0 = -d$

$n = \langle a, b, c \rangle$



$$m = \overrightarrow{P_0 P} = \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$\left| \text{comp}_n m \right| = \left| \frac{m \cdot n}{|n|} \right| = \left| \frac{a(x - x_0) + b(y - y_0) + c(z - z_0)}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$= \left| \frac{ax + by + cz - (ax_0 + by_0 + cz_0)}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$\text{distance} = \frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example: Find the distance between the point $(3, -2, 7)$ and the plane $4x - 6y + z = 5$

$$4x - 6y + z - 5 = 0$$

$$n = \langle 4, -6, 1 \rangle$$

$$\text{distance} = \frac{|4(3) - 6(-2) + 7 - 5|}{\sqrt{16 + 36 + 1}} = \frac{26}{\sqrt{53}}$$