Section 13.1: Vector Functions and Space curves

Let $\mathbf{r}$ be a **vector function** whose domain is a set of real numbers and result is a three-dimensional vector. Let

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where $f(t)$, $g(t)$, and $h(t)$ are real valued functions and are called the component functions of $\mathbf{r}$.

The limit of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions:

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

A vector function $\mathbf{r}$ is continuous if and only if its component functions $f(t)$, $g(t)$, and $h(t)$ are continuous.

Example: Given $\mathbf{r}(t) = \left\langle t\sqrt{t+5},\ t^2+2,\ \dfrac{e^t-1}{t} \right\rangle$

a) Find the domain of $\mathbf{r}(t)$.

$f = t\sqrt{t+5}$ $\quad t \geq -5$

$g = t^2+2$ $\quad$ all reals.

$h = \dfrac{e^t-1}{t}$ $\quad t \neq 0$

$[-5, 0) \cup (0, \infty)$

b) Find all t where $\mathbf{r}(t)$ is continuous.

$[-5, 0) \cup (0, \infty)$

note: at $t = -5$ we have Right continuity

c) Compute $\lim\limits_{t\to 0} \mathbf{r}(t)$. $= \left\langle \lim\limits_{t\to 0} t\sqrt{t+5},\ \lim\limits_{t\to 0} t^2+2,\ \underbrace{\lim\limits_{t\to 0} \dfrac{e^t-1}{t}} \right\rangle$

$= \left\langle 0,\ 2,\ \lim\limits_{t\to 0} \dfrac{e^t}{1} \right\rangle$

use L'Hopital

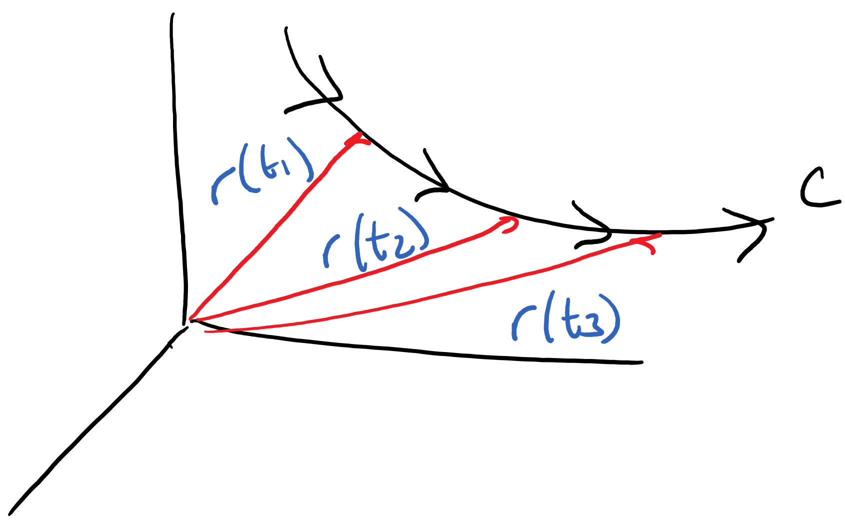
$= \langle 0, 2, 1 \rangle$

Definition: Suppose that $f(t)$, $g(t)$, and $h(t)$ are real valued functions on an interval I, then the set C defined as :

$$C = \{(x, y, z) | x = f(t), y = g(t), z = h(t)\}$$

where t is a parameter and t varies in some interval, I, is called a **space curve**. The space curve C can be traversed by the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.

$t_1 < t_2 < t_3$

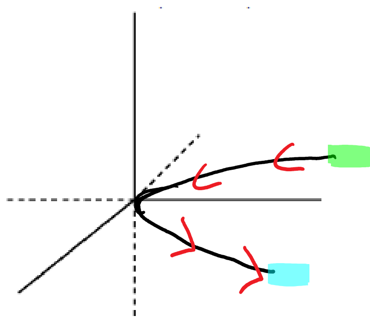


Example: Describe the curve defined by the vector function. Indicate the direction of motion.

(a) $\mathbf{r}(t) = \langle t, t^2, 0 \rangle$

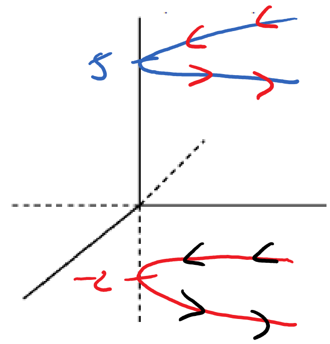
$$\left.\begin{array}{l} x = t \\ y = t^2 \end{array}\right\} \to y = x^2$$

$z = 0$



(b) $\mathbf{r}(t) = \langle t, t^2, \underline{c} \rangle$, where c is a constant.

if $c = 5$



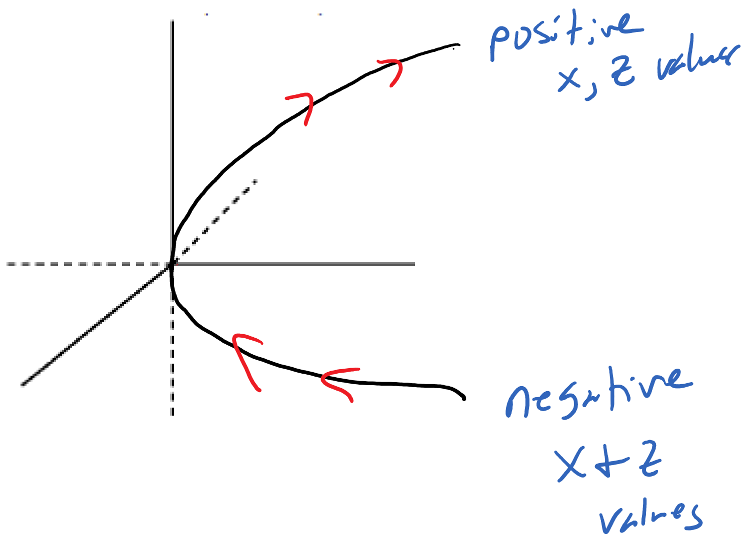
if $\underbrace{c = -2}_{\text{plane } z = -2}$

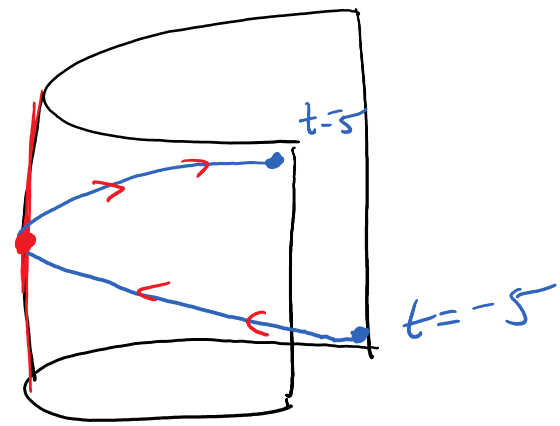
(c) $\mathbf{r}(t) = \langle t, t^2, t \rangle$.

$x = t$

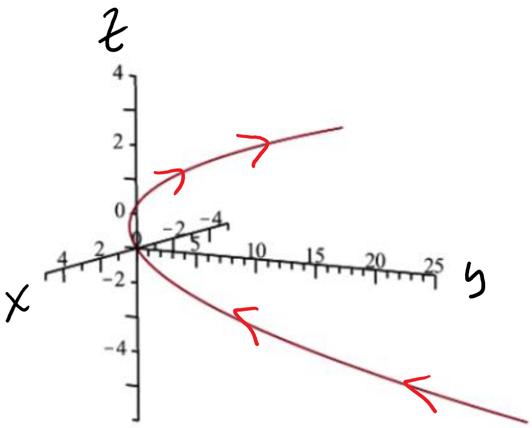
$y = t^2$ $\Big] \to y = x^2$

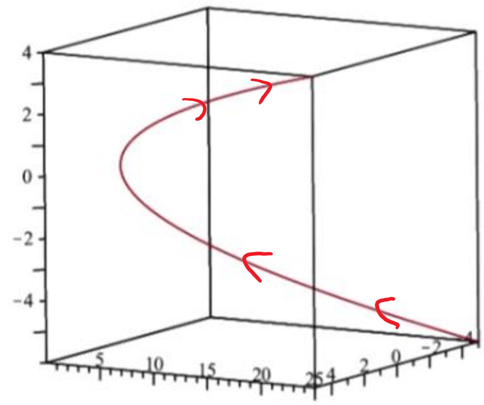
$z = t$

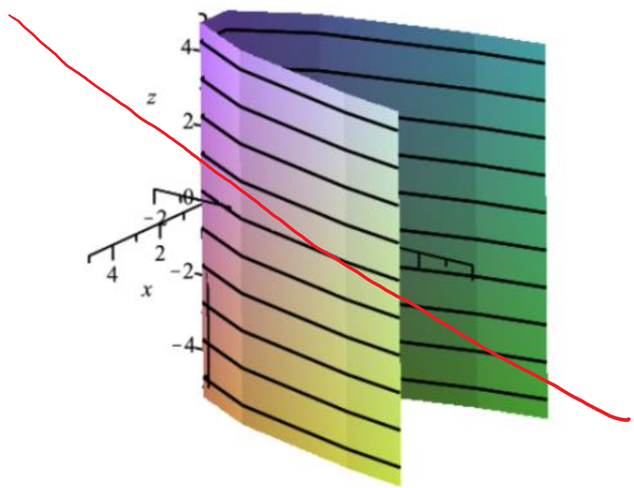




graphs



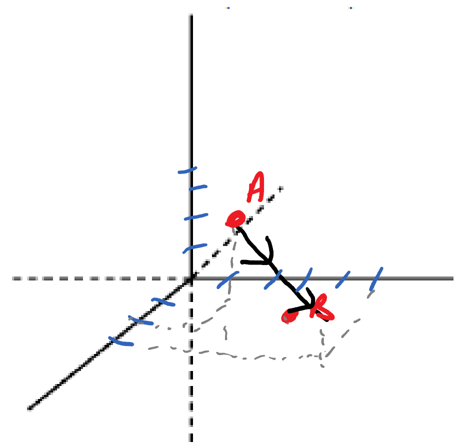




$$\left.\begin{array}{l} x=t \\ z=t \end{array}\right\} \to \underline{\underline{x=z}}$$

Pg 6

(d) $\mathbf{r}(t) = \langle 2+t, 2+3t, 4-2t \rangle$, $0 \leq t \leq 1$



$t=0 \quad r(0) = \langle 2, 2, 4 \rangle$

point $(2,2,4)$ A

$t=1 \quad r(1) = \langle 3, 5, 2 \rangle$

point $(3,5,2)$ B

Example: Show that the curve $\mathbf{r}(t) = \left\langle \sin(t), 2\cos(t), \sqrt{3}\sin(t) \right\rangle$ lies on both a plane and a sphere. What does the space curve for $\mathbf{r}(t)$ look like?

$$\left.\begin{aligned} x &= \sin(t) \\ y &= 2\cos(t) \\ z &= \sqrt{3}\sin(t) \end{aligned}\right] \longrightarrow \underbrace{z = \sqrt{3}\,x}_{} \longrightarrow \text{plane.}$$

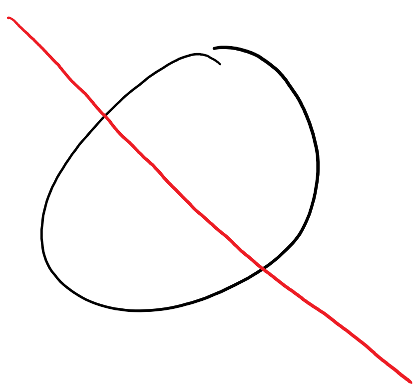
$$x^2 + y^2 + z^2 = \sin^2(t) + (2\cos(t))^2 + (\sqrt{3}\sin(t))^2$$

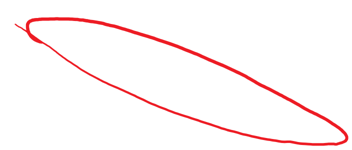
$$= \sin^2 t + 4\cos^2(t) + 3\sin^2(t)$$

$$= 4\sin^2 t + 4\cos^2 t$$

$$= 4\left[\sin^2(t) + \cos^2(t)\right]$$

$$x^2 + y^2 + z^2 = 4$$





$x = \sin t$

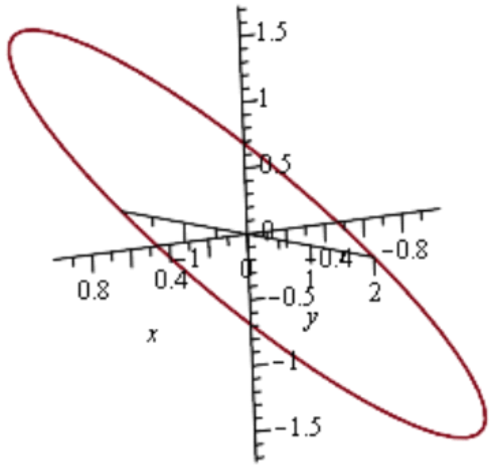
$y = 2\cos t$

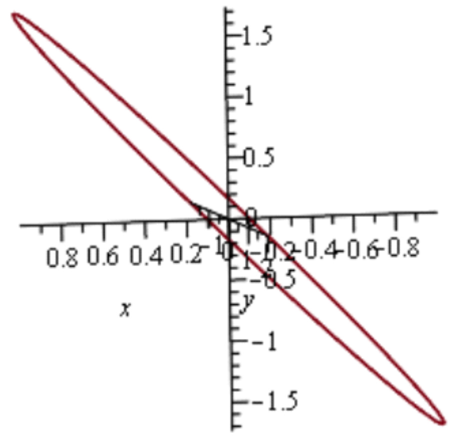
$$\sin^2 t + \cos^2 t = 1$$

$$x^2 + \left(\frac{y}{2}\right)^2 = 1$$

elliptical cylinder

Graphs 2





y-axis is comming out of the screen.

Example: Find a vector function that represents the curve of intersection of the two surfaces.
$x^2 + y^2 = 4$ and $z = xy$

$x = 2\cos t$

$y = 2\sin t$

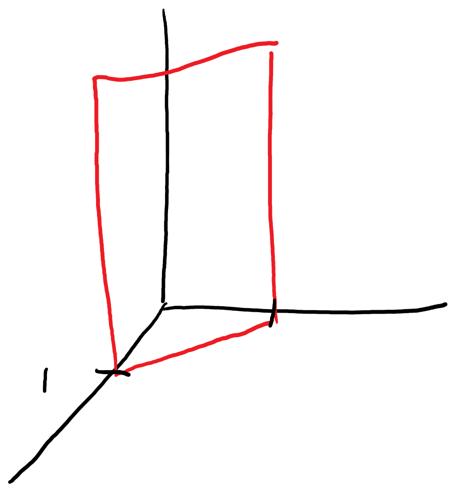
$z = 4\cos t \sin(t)$

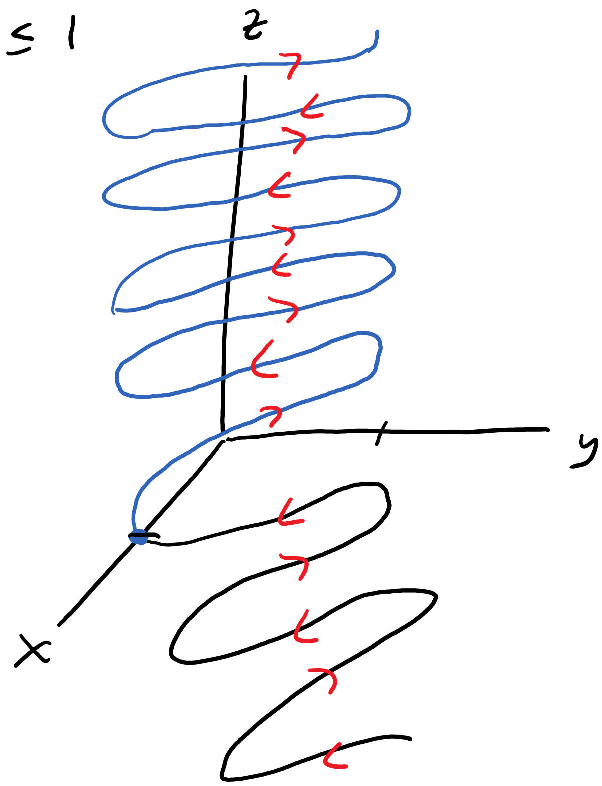
Example: Sketch the curve $x = \cos^2 t$, $y = \sin^2 t$, and $z = t$.

$x + y = 1$

$0 \le x \le 1$

$0 \le y \le 1$

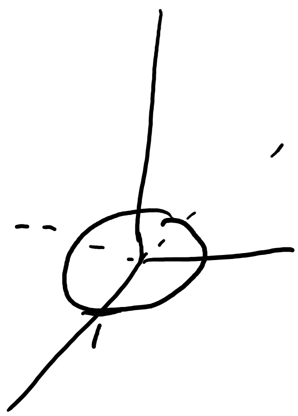




$x = \cos t$

$y = \sin t$

$z = t$



$x^2 + y^2 = 1$