Section 14.6: Directional Derivatives and the Gradient Vector

Recall that for $f(x, y)$, the first partial f_x represent the rate of change of f in the x direction and f_y represents the rate of change of f in the y direction. In other words, f_x and f_y represent the rate of change of f in the direction of the unit vectors **i** and **j** respectively.

Definition: The **directional derivative** of f at (x_o, y_o) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$, denoted by $D_{\mathbf{u}} f(x_o, y_o)$, is

$$D_{\mathbf{u}} f(x_o, y_o) = \lim_{h \to 0} \frac{f(x_o + ha, y_o + hb) - f(x_o, y_o)}{h}$$

if this limit exists.

Note: This shows that $f_x(x_o, y_o) = D_{\mathbf{i}} f(x_o, y_o)$ and $f_y(x_o, y_o) = D_{\mathbf{j}} f(x_o, y_o)$.

Theorem: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Note: If the unit vector $\mathbf{u}$ makes an angle θ with respect to the positive x–axis, then we can write

$\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and $D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$

Example: Find the rate of change of $f(x, y) = x^2 + 2xy - 3y^2$ at the point $(1, 2)$ in the direction indicated by the angle $\theta = \dfrac{\pi}{4}$.

$u = \langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$f_x = 2x + 2y$ $\qquad f_x(1,2) = 6$

$f_y = 2x - 6y$ $\qquad f_y(1,2) = -10$

$D_u f(x,y) = 6 \cdot \frac{\sqrt{2}}{2} + -10 \frac{\sqrt{2}}{2} = 3\sqrt{2} - 5\sqrt{2}$

$= -2\sqrt{2}$

Definition: If f is a function of two variables x and y, then the **gradient** of f, denoted **grad** f or ∇f, is the vector function defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \langle f_x, f_y \rangle$$

Note: ∇f which is read "del f".

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$$

Theorem: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x, f_y \rangle \cdot \langle a, b \rangle$$

$$= \nabla f(x,y) \cdot u$$

Definition: The gradient and the directional derivative of a function $f(x, y, z)$ with unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$ $\qquad$ $D_{\mathbf{u}}f(x, y, z) = \nabla f(x,y,z) \cdot u$

Example: Find the gradient and the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = \langle 2, 5 \rangle$.

gradient function $\nabla f = \langle 2xy^3, 3x^2y^2 - 4 \rangle$

$\nabla f(2,-1) = \langle -4, 8 \rangle$

$|v| = \sqrt{4+25}$
$= \sqrt{29}$

unit $= \frac{1}{\sqrt{29}} \langle 2, 5 \rangle$
$= \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle$

$D_u f(2,-1) = \nabla f(2,-1) \cdot u$
$= \langle -4, 8 \rangle \cdot \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle$
$= -4 \cdot \frac{2}{\sqrt{29}} + 8 \cdot \frac{5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$

Example: Find the directional derivative of the function $f(x, y, z) = z^4 - x^3y^2$ at the point $P(1, 3, 2)$ in the direction of $Q(2, 4, 3)$.

$$\vec{PQ} = \langle 1, 1, 1 \rangle$$

$$\text{unit} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$$

$$\nabla f = \langle -3x^2y^2, -2x^3y, 4z^3 \rangle$$

$$\nabla f(1,3,2) = \langle -27, -6, 32 \rangle$$

$$D_u f(1,3,2) = \langle -27, -6, 32 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$$

$$= \frac{1}{\sqrt{2}} \left[-27(1) - 6(1) + 32 \right]$$

$$= \frac{1}{\sqrt{3}} (-1) = \frac{-1}{\sqrt{3}}$$

The directional derivatives at a point P for a function f gives the rates of changes of f in all possible directions. This leads to the question: In which of these directions does f change the fastest and what is the maximum rate of change?

Theorem: Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occur when **u** has the direction as the gradient vector.

$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|u| \cos\theta \longrightarrow |u| = 1$

$= |\nabla f| \cos\theta$ if $\theta = 0$ then $\cos\theta = 1$

max $D_u f = |\nabla f|$

min $D_u f = -|\nabla f|$ when $\theta = \pi$

Example: If $f(x, y) = xe^y$, in what direction does f have the maximum rate of change at the point $P(2, 0)$? What is the maximum rate of change of f?

$$\nabla f = \langle e^y, xe^y \rangle$$

$$\nabla f(2,0) = \langle e^0, 2e^0 \rangle = \boxed{\langle 1, 2 \rangle}$$

$$\text{Max Rate of change} = |\nabla f(2,0)| = \sqrt{1^2 + 2^2}$$

$$= \boxed{\sqrt{5}}$$

Tangent Planes to Level surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. Let $P(x_o, y_o, z_o)$ be a point on the surface.

Let C be any curve on the surface going though the point P and defined by the vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ with $\mathbf{r}(t_o) = \langle x_o, y_o, z_o\rangle$.

Combining the above information gives $F(x(t), y(t), z(t)) = k$. Now if F and $\mathbf{r}$ are differentiable, then we can use the chain rule to get the following.

$$F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt} = 0$$

$$\langle F_x, F_y, F_z\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle = 0$$

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

Thus ∇F at point P is perpendicular to the tangent vector $\mathbf{r}'(t_o)$ to any curve C on the surface S passing though the point P.

This means that ∇F at point P is a normal vector for the tangent plane.

Thus the tangent plane to the surface $F(x,y,z)=k$ at point $P(x_o,y_o,z_o)$ is

$$F_x(x_o,y_o,z_o)(x-x_o)+F_y(x_o,y_o,z_o)(y-y_o)+F_z(x_o,y_o,z_o)(z-z_o)=0$$

$F_x\ (x-x_0)+F_y\ (y-y_0)+F_z\ (z-z_0)=0$

$F(x,y,z)=k$

$n=\langle F_x, F_y, F_z\rangle$

What is a direction vector for the normal line to the surface at the point $P(x_o,y_o,z_o)$?

14.4

$n=\langle f_x, f_y, -1\rangle$

$z=f(x,y)$

normal line

$x=x_0+tF_x$

$y=y_0+tF_y$

$z=z_0+tF_z$

Example: Find the equation of the tangent plane and the normal line to the surface at the point $(1, 2, 1)$

$$\underbrace{4x^2 + y^2 + 9z^2}_{F(x,y,z)} = 17$$

$\nabla F = \langle 8x, 2y, 18z \rangle$

$\nabla F(1,2,1) = \langle 8, 4, 18 \rangle$

Tangent plane

$8(x-1) + 4(y-2) + 18(z-1) = 0$

normal line

$x = 1 + 8t$

$y = 2 + 4t$

$z = 1 + 18t$

In a similar manner, the gradient vector $\nabla F(x_o, y_o)$ is perpendicular to the level curves $f(x, y) = k$ at the point (x_o, y_o).

Consider the topographical map of a hill and let $f(x, y)$ represent the elevation at the point (x, y). Draw a curve of steepest ascent.