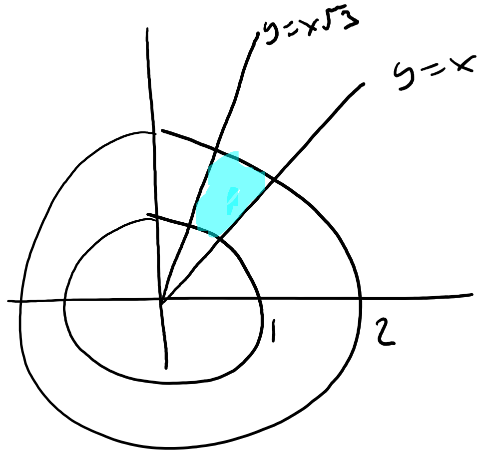
Example: Evaluate $\iint\limits_D \arctan\left(\frac{y}{x}\right) dA$

where $D = \{(x, y) \mid 1 \le x^2 + y^2 \le 4,\ x \le y \le x\sqrt{3},\ x \ge 0\}$.

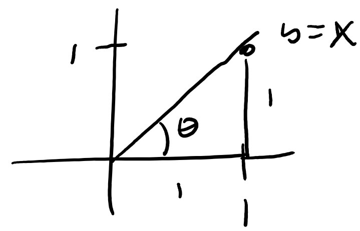


$y = x\sqrt{3}$

$\tan\theta = \frac{\sqrt{3}}{1}$

$\theta = \frac{\pi}{3}$

$y = x \to \theta = \frac{\pi}{4}$



$\tan\theta = \frac{1}{1}$

$\theta = \frac{\pi}{4}$

$1 \le r \le 2 \qquad \frac{\pi}{4} \le \theta \le \frac{\pi}{3}$

$$\arctan\frac{y}{x} = \arctan\left(\frac{r\sin\theta}{r\cos\theta}\right) = \arctan\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$= \arctan(\tan\theta) = \theta \quad \text{if } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\iint\limits_D \arctan\left(\frac{y}{x}\right) dA = \int_{\theta=\pi/4}^{\pi/3} \int_{r=1}^{2} \theta \cdot r \, dr \, d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\pi/3} \theta \, d\theta \cdot \int_{r=1}^{2} r \, dr$$

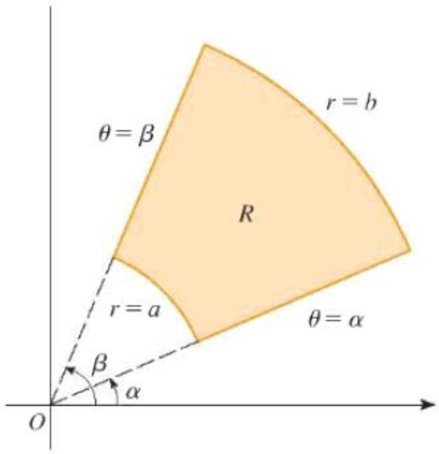
$$= \frac{1}{2}\theta^2 \Big|^{\pi/3} \cdot \frac{1}{2} r^2 \Big|^{2}$$

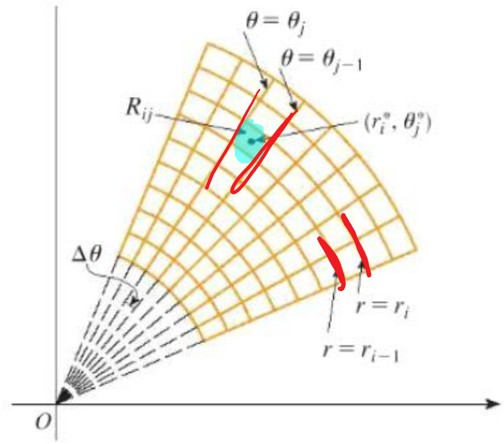
$$= \frac{1}{2}\theta^2 \Big|_{\pi/4}^{\pi/3} \cdot \frac{1}{2} r^2 \Big|_{1}^{2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{3}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] \cdot \frac{1}{2}\left[4 - 1\right]$$

$$= \frac{7\pi^2}{192}$$

Pg 2: Polar integral derivation

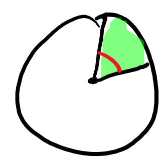




The center of the polar subrectangle $R_{ij} = \{(r,\theta)|r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$

has polar coordinates: $r_i^* = \frac{1}{2}(r_{i-1} + r_i)$ and $\theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$

Note: the area a sector of a circle with radius r and angle θ is $\dfrac{1}{2}r^2\theta$.



$$\Delta A_{ij} = \frac{1}{2}r_i^2\Delta\theta_j - \frac{1}{2}r_{i-1}^2\Delta\theta_j = \frac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\Delta\theta_j$$

$$\Delta A_{ij} = \frac{1}{2}\left(r_i + r_{i-1}\right)\left(r_i - r_{i-1}\right)\Delta\theta_j = r_i^*\Delta r_i\Delta\theta_j$$

$\Delta A = r^* \Delta r\ \Delta\theta$

thus $dA = r\ drd\theta$

Change to Polar Coordinates in a Double Integral: If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $c \le \theta \le d$, where $0 \le d - c \le 2\pi$, then

$x = r\cos\theta$
$y = r\sin\theta$

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(r\cos\theta, r\sin\theta)\ r\ drd\theta$$

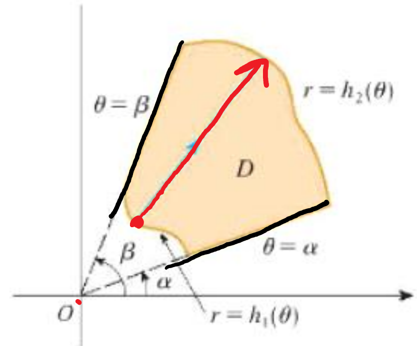
Pg 3: non-rectangular regions

If f is continuous on a polar region of the form

$D = \{(r, \theta) | c \leq \theta \leq d, h_1(\theta) \leq r \leq h_2(\theta)\}$

then

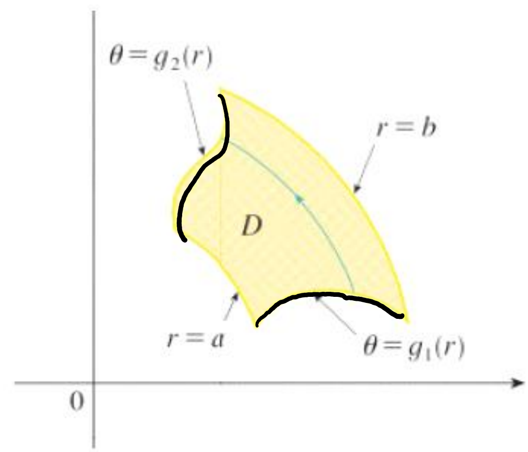
$$\iint_D f(x,y)dA = \int_{\theta = c}^{d} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\ r dr d\theta$$



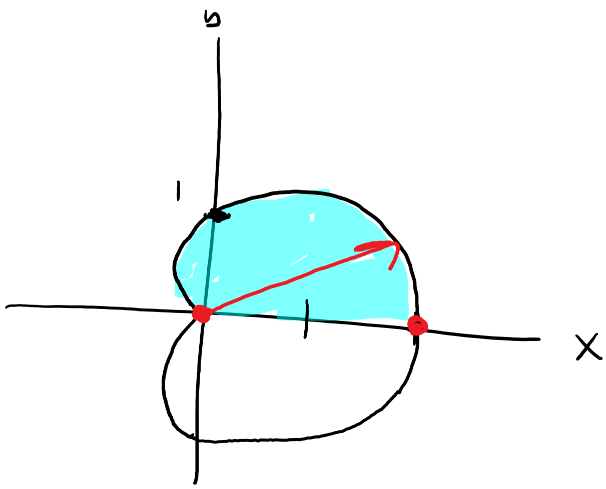
If $D = \{(r, \theta) | a \leq r \leq b, g_1(r) \leq \theta \leq g_2(r)\}$

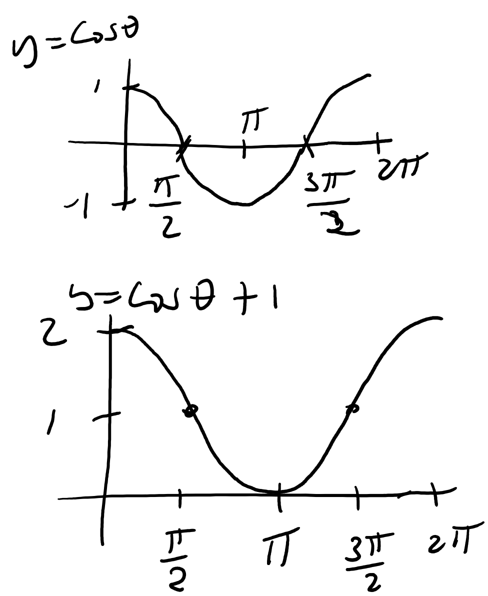
then

$$\iint_D f(x,y)dA = \int_{r = a}^{b} \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)\ r d\theta dr$$



Example: Compute $\iint\limits_D y\,dA$ where D is the upper half of the cardioid: $r = 1+\cos\theta$.





$x = r\cos\theta$

$y = r\sin\theta$

$0 \le \theta \le \pi$ $\qquad$ $0 \le r \le 1+\cos\theta$

$$\iint\limits_D y\,dA = \int_{\theta=0}^{\pi}\int_{r=0}^{1+\cos\theta} r\sin\theta \cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{1+\cos\theta} r^2\sin\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi} \frac{1}{3}r^3\sin\theta \Big|_{r=0}^{1+\cos\theta} d\theta$$

$$= \int^{\pi} \frac{1}{3}(1+\cos\theta)^3 \underline{\sin\theta\,d\theta}$$

$u = 1+\cos\theta$

$du = -\sin\theta\,d\theta$

$-du = \sin\theta \, d\theta$

$\theta = \pi$

$u = 0$

$\theta = 0$

$u = 2$

$$= \int_{u=2}^{0} -\frac{1}{3} u^3 \, du$$

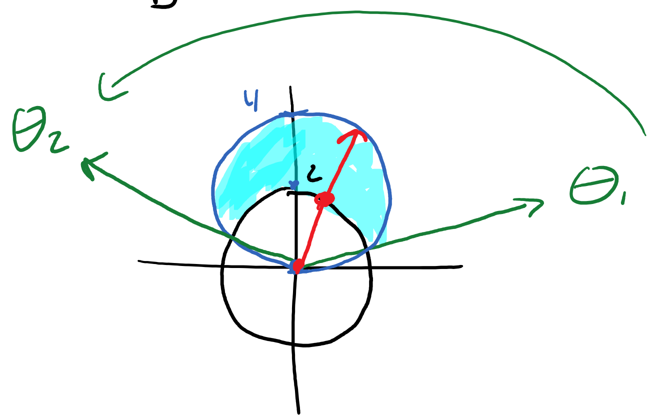
$$= -\frac{1}{12} u^4 \Big|_{u=2}^{0} = -\frac{1}{12}(0)^4 - \left(-\frac{1}{12}(2)^4\right)$$

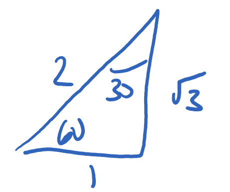
$$= 0 + \frac{1}{12} \cdot 16$$

$$= \frac{16}{12}$$

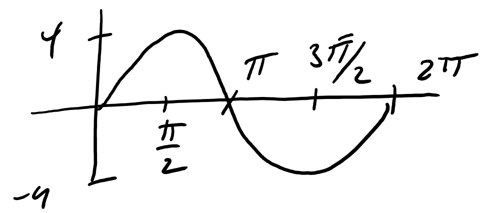
Example: Find the area of the region inside the circle $r = 4\sin\theta$ and out side the circle $r = 2$.

$$\text{Area} = \iint_D 1\, dA$$





$r = 4\sin\theta$



$$4\sin\theta = 2$$
$$\sin\theta = \frac{1}{2}$$
$$\theta_1 = \frac{\pi}{6}$$
$$\theta_2 = \frac{5\pi}{6}$$

$$r^2 = 4r\sin\theta$$
$$x^2 + y^2 = 4y$$

$$\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$$
$$2 \le r \le 4\sin\theta$$

$$\iint_D 1\, dA = \int_{\theta=\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{r=2}^{4\sin\theta} 1 \cdot r\, dr\, d\theta$$

$$= 2 \int_{\theta=\frac{\pi}{6}}^{\pi/2} \int_{r=2}^{4\sin\theta} 1\, r\, dr\, d\theta$$

$$= 2 \int^{\pi/2} \frac{1}{2} r^2 \Big|_{r=2}^{4\sin\theta} d\theta$$

$\theta = \frac{\pi}{6}$ $\qquad$ $r = 2$

$$= \int_{\theta=\frac{\pi}{6}}^{\pi/2} (4\sin\theta)^2 - (2)^2 \, d\theta = \int_{\theta=\frac{\pi}{6}}^{\pi/2} 16 \sin^2\theta - 4 \, d\theta$$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$= \int_{\theta=\frac{\pi}{6}}^{\pi/2} 16\left[\frac{1}{2}(1 - \cos 2\theta)\right] - 4 \, d\theta$$

$$= \int_{\theta=\frac{\pi}{6}}^{\pi/2} 8 - 8\cos 2\theta - 4 \, d\theta = \int_{\theta=\frac{\pi}{6}}^{\pi/2} 4 - 8\cos 2\theta \, d\theta$$

$$= \left(4\theta - 8 \cdot \frac{1}{2} \sin(2\theta)\right)\Bigg|_{\theta=\pi/6}^{\pi/2}$$

$$= 2\pi - 4\sin(\pi) - \left[\frac{4\pi}{6} - 4\sin\left(\frac{\pi}{3}\right)\right]$$

$$= 2\pi - 0 - \frac{4\pi}{6} + 4\,\frac{\sqrt{3}}{2}$$

$= 2\pi \quad 2\pi \quad 2\sqrt{\ }$

$$\frac{}{3} + 2\sqrt{3}$$

$$= \frac{4\pi}{3} + 2\sqrt{3}$$

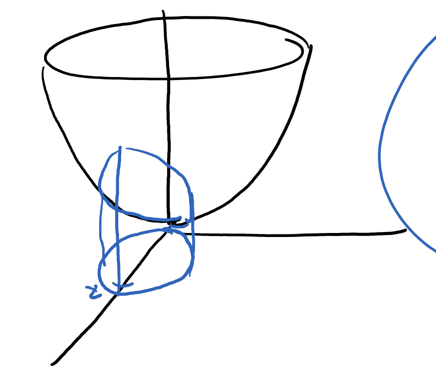
Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy–plane and inside the cylinder $x^2 + y^2 = 2x$.

$$x^2 - 2x + y^2 = 0$$

$$x^2 - 2x + (-1)^2 + y^2 = (-1)^2$$

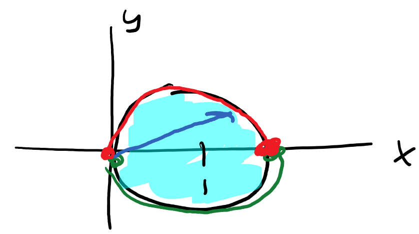
$$(x-1)^2 + y^2 = 1$$

$$V = \iint_D x^2 + y^2 \, dA$$



$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

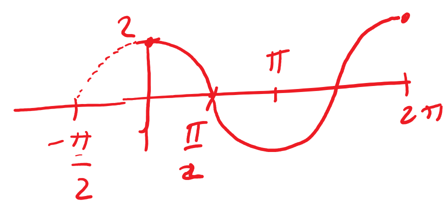
$$0 \leq r \leq 2\cos\theta$$



$$x^2 + y^2 = 2x$$

$$r^2 = 2r\cos\theta$$

$$r = 2\cos\theta$$



$$V = \iint_D x^2 + y^2 \, dA = \int_{\theta = -\frac{\pi}{2}}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2 \cdot r \, dr \, d\theta$$

$$= \int_{\theta = -\frac{\pi}{2}}^{\pi/2} \int_{r=0}^{2\cos\theta} r^3 \, dr \, d\theta$$

$$= \int^{\pi/2} \frac{1}{\cdot} r^4 \Big|^{2\cos\theta} d\theta = \int^{\pi/2} \frac{1}{\cdot} \cdot 16\cos^4\theta \, d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \frac{1}{4} r^4 \Big|_0^{2\cos\theta} d\theta = \int_{\theta=-\frac{\pi}{2}} \frac{1}{4}\cdot 16\cos^4\theta \, d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} 4\cos^4\theta \, d\theta = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} 4\cdot\cos^2\theta\cos^2\theta \, d\theta$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} 4\cdot\frac{1}{2}(1+\cos 2\theta)\frac{1}{2}(1+\cos 2\theta) \, d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta \, d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} 1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta) \, d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta \, d\theta$$

$$\Big|^{\pi/2}$$

$$\theta = \frac{\quad}{2}$$

$$= \frac{3}{2}\theta + 2\cdot\frac{1}{2}\sin 2\theta + \frac{1}{2}\cdot\frac{1}{4}\sin(4\theta)\Bigg|_{-\frac{\pi}{2}}^{\quad/2}$$

$$= \frac{3}{2}\left(\frac{\pi}{2}\right) + \sin(\pi) + \frac{1}{8}\sin(2\pi) - \left[-\frac{3\pi}{4} + \sin(-\pi) + \frac{1}{8}\sin(-2\pi)\right]$$

$$= \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$