Section 15.4: Applications of Double Integrals

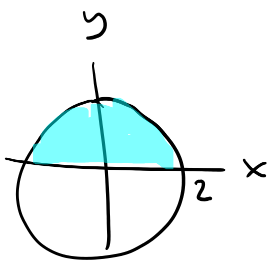
- **Area of a region D:** $\iint\limits_D 1\, dA = A(D)$ the area of region D.
- **Volume of a solid** with base D and $f(x, y) \geq 0$ on region D: $\iint\limits_D f(x, y) dA$
- The **total mass** of a lamina (thin plate or region) with variable density (in units of mass per unit area) on the region D is given by

  $$m = \iint\limits_D \rho(x, y) dA$$

  where $\rho(x, y)$ is the density at the point (x, y).

  Note: If the density is constant then $m = \rho *$ Area of D.

Example: An electric charge is distributed over the part of the disk $x^2 + y^2 \leq 4$ in the top half of the xy–plane so that the charge density at (x, y) is $\sigma(x, y) = x^2 + y^2$ (measured in coulombs per square meter). Find the total charge on the disk.



$0 \leq r \leq 2$

$0 \leq \theta \leq \pi$

$$\text{Total charge} = \iint_D \sigma(x,y)\, dA$$

$$= \iint_D x^2+y^2\, dA$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{2} r^2 \cdot r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{2} r^3\, dr\, d\theta$$

$$= \int_0^{\pi} 1\, d\theta \cdot \int_{r=0}^{2} r^3\, dr$$

$-2 \leq x \leq 2$

$0 \leq y \leq \sqrt{4-x^2}$

$$\int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} x^2+y^2\, dy\, dx$$

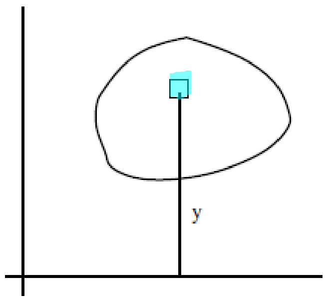
$$= \theta \Big|_0^{\pi} \cdot \frac{1}{4} r^4 \Big|_0^{2} = (\pi - 0) \cdot \left(\frac{1}{4}(2)^4 - 0\right)$$

$$= \pi \cdot \frac{16}{4} = 4\pi$$

md = moment

Moments and Center of Mass

The moment of a particle about an axis is defined to be the mass of the particle times the distance of the particle from the axis.



$$M_x = \sum_i \sum_j \left(\rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij} \quad y_{ij}\right)$$

$$M_y = \sum \sum \left(\rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij} \quad x_{ij}\right)$$

Given a lamina with density function $\rho(x, y)$. The moment of the lamina about the $x-$axis, denoted M_x, and the moment about the $y-$axis, denoted M_y, is given by

$$M_x = \iint_D y\rho(x, y)dA \qquad M_y = \iint_D x\rho(x, y)dA$$

$m\bar{x} = M_y$

$m\bar{y} = M_x$

The **center of mass**, $(\overline{x}, \overline{y})$ of the lamina with density $\rho(x, y)$ are

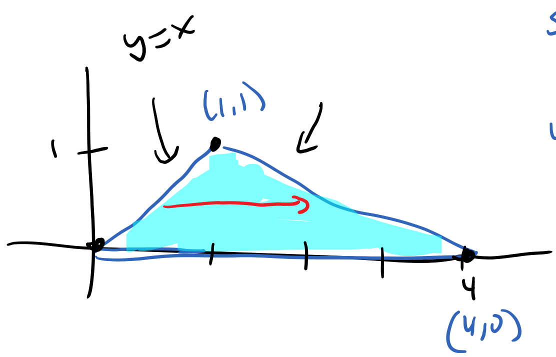
$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)dA \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)dA \qquad \text{where } m = \iint_D \rho(x, y)dA$$

Moments of Inertia: (second moment).

about the x-axis: $I_x = \displaystyle\iint_D y^2\rho(x, y)dA$ about the y-axis: $I_y = \displaystyle\iint_D x^2\rho(x, y)dA$

about the origin: $I_o = I_x + I_y = \displaystyle\iint_D (x^2 + y^2)\rho(x, y)dA$

Example: Find the mass of the lamina occupies the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(4, 0)$. The density of the region is given by $\rho(x, y) = y$. Note the center of mass calculations can be found with the additional problems.



$\text{slope} = \frac{1-0}{1-4} = \frac{-1}{3}$

$y - 0 = \frac{-1}{3}(x-4)$

$y = -\frac{x}{3} + \frac{4}{3}$

$3y = -x + 4$

$x = 4 - 3y$

$\text{mass} = \iint_D \rho(x,y)\, dA$

$= \iint_D y\, dA$

~~polar~~ X

$dy\,dx \uparrow$

$\boxed{dx\,dy \rightarrow}$

$0 \le y \le 1$

$y \le x \le 4 - 3y$

$$\int_{y=0}^{1} \int_{x=y}^{4-3y} y\, dx\, dy = \int_{y=0}^{1} xy \Big|_{x=y}^{4-3y} dy$$

$$= \int_{y=0}^{1} (4-3y)y - y^2\, dy = \int_{y=0}^{1} 4y - 3y^2 - y^2\, dy$$

$$= \int_{y=0}^{1} 4y - 4y^2\, dy = \left(2y^2 - \frac{4y^3}{3}\right)\Big|_0^1$$

$$= 2 - \frac{4}{3} - (0) = \frac{6}{3} - \frac{4}{3} = \frac{2}{3}$$

$$M_x = \iint_D y\,\rho(x,y)\,dA = \int_{y=0}^{1}\int_{x=y}^{4-3y} y \cdot y \; dx\,dy$$

$$M_y = \iint_D x\,\rho(x,y)\,dA = \int_{y=0}^{1}\int_{x=y}^{4-3y} xy \; dx\,dy$$