**Cylindrical Coordinates:**
A Cartesian point $(x, y, z)$ is represented by $(r, \theta, z)$ in the Cylindrical Coordinate System. Where $(r, \theta)$ represent the polar coordinates for the point $(x, y)$ and $z$ is the distance above or below the $xy$-plane.

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$ $\quad$ $z = z$ $\quad$ $r^2 = x^2 + y^2$ $\quad$ $\tan\theta = \frac{y}{x}$

Note: Cylindrical coordinates are useful in problems that involve symmetry about the $z$-axis.

Example: Find the cylindrical coordinates for the point $(-1, \sqrt{3}, 2)$

$x \ y \ z$

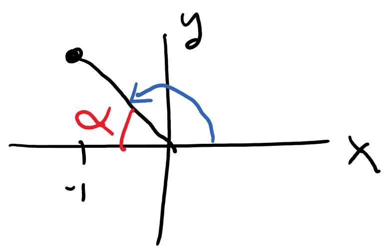
$r = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$= \sqrt{1+3}$

$r = \sqrt{4} = 2$

$\tan\theta = \frac{\sqrt{3}}{-1}$

$\tan\theta = -\frac{\sqrt{3}}{1}$




Reference angle

$\tan\alpha = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\alpha = 60^\circ = \frac{\pi}{3}$

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$$\begin{array}{ccc} x & y & z \\ (-1, & \sqrt{3}, & 2) \end{array} \longrightarrow \begin{array}{ccc} r & \theta & z \\ (2, & \frac{2\pi}{3}, & 2) \end{array}$$

Example: Write the equations in cylindrical coordinates.

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

A) $z = 12 - 4x^2 - 4y^2$

$= 12 - 4(x^2 + y^2)$

$z = 12 - 4r^2$

cone

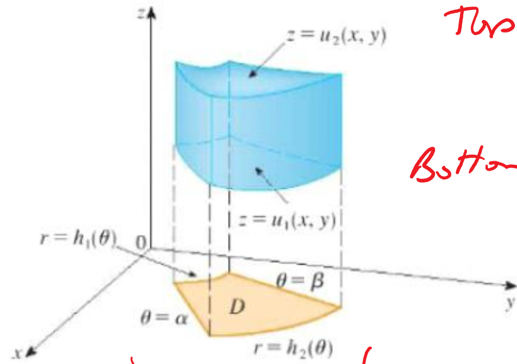
B) $z = \sqrt{3x^2 + 3y^2}$ $= \sqrt{3(x^2 + y^2)}$

$z = \sqrt{3r^2}$

$z = \sqrt{3}\, r$

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[ \int_z f(x,y,z)\,dz \right] dA$$

### Triple Integrals in Cylindrical Coordinates




Top

Bottom

Suppose that E is a solid whose image D on the $xy$−plane can be described in polar coordinates.

$E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$

and $D = \{(r, \theta)|a \le \theta \le b, h1(\theta) \le r \le h2(\theta)\}$

If $f(x, y, z)$ is continuous over the solid E,
$g_1(r, \theta) = u_1(r\cos\theta, r\sin\theta)$, and
$g_2(r, \theta) = u_2(r\cos\theta, r\sin\theta)$ then

$$\iiint_E f(x,y,z)dV = \int_{\theta=a}^{b} \int_{r=h_1(\theta)}^{h_2(\theta)} \int_{z=g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\ r\ dzdrd\theta$$

Example A solid lies with in the cylinder $x^2 + y^2 = 4$, below the plane $z = 8$, and above the paraboloid $z = 4 - x^2 - y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

$\rho(x,y,z) = K\sqrt{x^2+y^2}$

$$\text{mass} = \iiint_E \rho(x,y,z)\, dV$$

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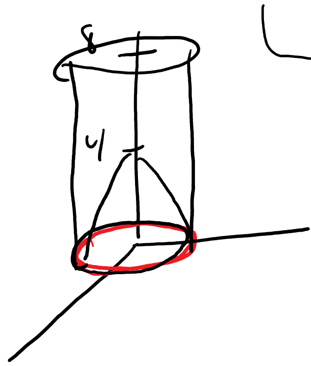
$x^2+y^2=4$

Top $z=8$ $\quad z=8$

Bottom $z=4-x^2-y^2$

$z=4-r^2$

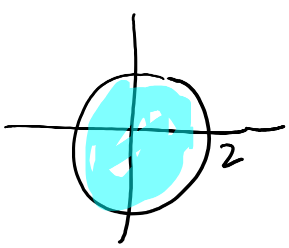



$z = 4-(x^2+y^2)$

$z = 4-4$

$z=0$

Region D




$0 \le r \le 2$

$0 \le \theta \le 2\pi$

density $= K\sqrt{r^2} = Kr$

$$\text{mass} = \int_{\theta=0}^{2\pi} \underbrace{\int_{r=0}^{2} \int_{z=4-r^2}^{8} Kr \cdot r \, dz\, dr}\, d\theta$$

$$= \int_{\theta=0}^{2\pi} 1\, d\theta \cdot \int_{r=0}^{2} \int_{z=4-r^2}^{8} Kr^2\, dz\, dr$$

$$= 2\pi \cdot \int_{r=0}^{2} K r^2 \cdot z \Big|_{z=4-r^2}^{8} \, dr$$

$$= 2\pi K \int_{r=0}^{2} r^2(8) - r^2(4-r^2) \, dr$$

$$= 2\pi K \int_{r=0}^{2} 8r^2 - 4r^2 + r^4 \, dr$$

$$= 2\pi K \int_{r=0}^{2} 4r^2 + r^4 \, dr = 2\pi K \left[ \frac{4r^3}{3} + \frac{r^5}{5} \right] \Bigg|_0^2$$

$$\boxed{2\pi K \left[ \frac{32}{3} + \frac{32}{5} \right]}$$

Example: Evaluate $\underbrace{\int_{x=0}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}}_{\text{Region } D} \underbrace{\int_{z=\sqrt{x^2+y^2}}^{2} \underbrace{(x^2+y^2)}\, dz}\, dy\, dx$

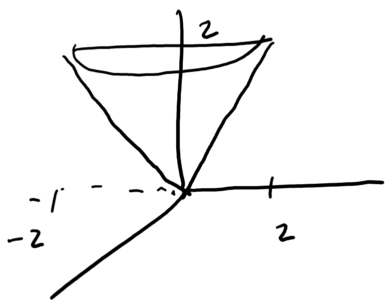
Top $z=2$

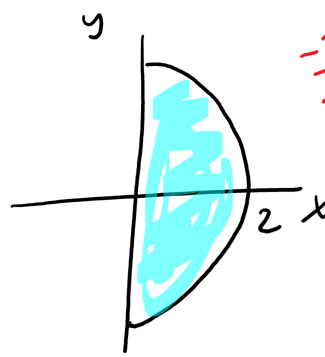
Bottom $z=\sqrt{x^2+y^2}=\sqrt{r^2}=r$

Region D

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$0 \le x \le 2$




$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 2$

$$\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\int_{r=0}^{2} \int_{z=r}^{2} r^2 \cdot r \, dz\, dr}\, d\theta$$

$$\int_{\theta=-\frac{\pi}{2}}^{\pi/2} 1\, d\theta \cdot \int_{r=0}^{2} \int_{z=r}^{2} r^3\, dz\, dr$$

$$= \pi \cdot \int_{r=0}^{2} r^3 \cdot z \Big|_{r}^{2} dr = \pi \int_{r=0}^{2} 2r^3 - r^4\, dr$$

$r=0$ $\qquad\qquad$ $r=0$

$$= \pi \left( \frac{2r^4}{4} - \frac{r^5}{5} \right) \Bigg|_0^2 = \pi \left( 8 - \frac{32}{5} \right)$$