Section 15.8: Triple Integrals in Spherical Coordinates

Spherical Coordinates:
A Cartesian point (x, y, z) is represented by (ρ, θ, ϕ) in the Spherical Coordinate System where $\rho \geq 0$ and $0 \leq \phi \leq \pi$.

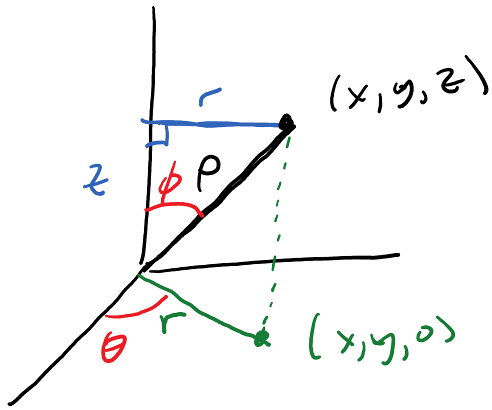
$$\rho = \sqrt{x^2+y^2+z^2}$$

$$r = \sqrt{x^2+y^2}$$

$$z = \rho\cos\phi$$

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\cos\phi = \frac{z}{\rho}$$

$$z = \rho\cos\phi$$

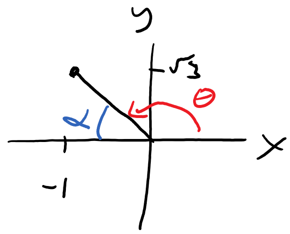
$$\sin\phi = \frac{r}{\rho}$$

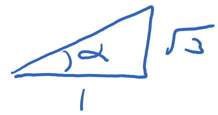
$$r = \rho\sin\phi$$

Example: Find the spherical coordinates for the points $(-1, \sqrt{3}, 2)$ and $(-1, \sqrt{3}, -2)$

$\rho = \sqrt{1+3+4}$

$\rho = \sqrt{8}$



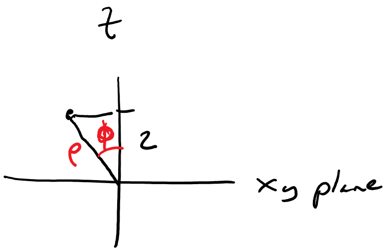


$\tan\alpha = \frac{\sqrt{3}}{1}$

$\tan\alpha = \sqrt{3}$

$\alpha = 60° = \frac{\pi}{3}$

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

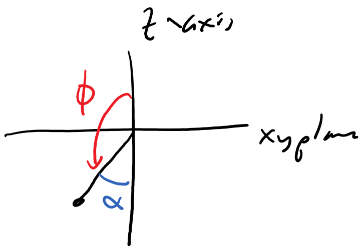


$\cos\phi = \frac{2}{\sqrt{8}} = \frac{2}{2\sqrt{2}}$

$\cos\phi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\phi = \frac{\pi}{4}$

$$\overset{x}{(-1},\ \overset{y}{\sqrt{3}},\ \overset{z}{2)} \longrightarrow \left(\overset{\rho}{\sqrt{8}},\ \overset{\theta}{\frac{2\pi}{3}},\ \overset{\phi}{\frac{\pi}{4}}\right)$$



$\alpha = \frac{\pi}{4}$

$\phi = \frac{3\pi}{4}$

$(-1, \sqrt{3}, -2)$?

$\left(\sqrt{8}, \frac{2\pi}{3}, \frac{3\pi}{4}\right)$

Example: Write the equations in spherical coordinates.

A) $x^2 + y^2 + z^2 = 25$

$$\rho^2 = 25$$
$$\rho = 5$$

B) $z = 12 - 4x^2 - 4y^2$

$$\rho \cos\phi = 12 - 4\rho^2 \sin^2\phi \cos^2\theta - 4\rho^2 \sin^2\phi \sin^2\theta$$
$$= 12 - 4\rho^2 \sin^2\phi \left[\cos^2\theta + \sin^2\theta\right]$$

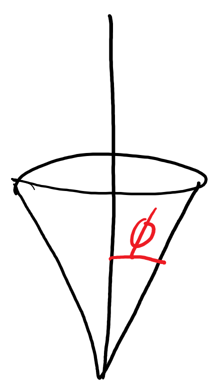
$$\boxed{\rho \cos\phi = 12 - 4\rho^2 \sin^2\phi}$$

C) $z = \sqrt{3x^2 + 3y^2}$ Cone.

$$\rho \cos\phi = \sqrt{3\rho^2 \sin^2\phi \cos^2\theta + 3\rho^2 \sin^2\phi \sin^2\theta}$$
$$= \sqrt{3\rho^2 \sin^2\phi \left[\cos^2\theta + \sin^2\theta\right]}$$

$$\rho \cos\phi = \sqrt{3\rho^2 \sin^2\phi}$$

$$\rho \cos\phi = \sqrt{3}\,\rho \sin\phi$$



$$\cos\phi = \sqrt{3} \sin\phi$$

$$\frac{1}{\sqrt{3}} = \tan\phi$$

$$\phi = \frac{\pi}{6}$$ Cone.

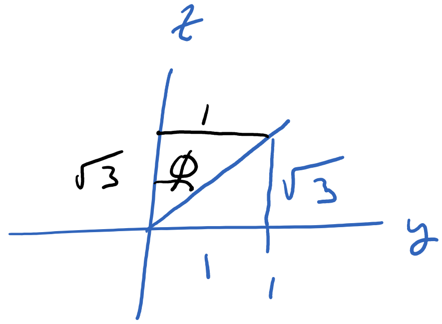
$z = \sqrt{3x^2 + 3y^2}$ cone

Lets look at a cross section. ie let $x=0$
or $y=0$

Let $x=0$

$z = \sqrt{0 + 3y^2}$

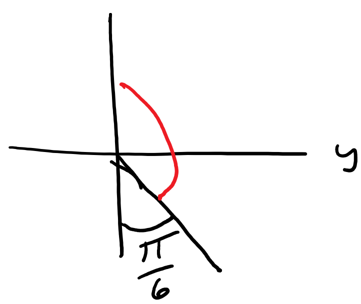
$z = \sqrt{3}\, y$



$\tan\phi = \frac{1}{\sqrt{3}}$

$\phi = \frac{\pi}{6}$

$z = -\sqrt{3x^2 + 3y^2}$



$\phi = \frac{5\pi}{6}$

Triple Integrals in Spherical Coordinates

In this coordinate system, the equivalent of a box is a spherical wedge
$E = \{(\rho, \theta, \phi) | a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$

where $a \ge 0$, $\beta - \alpha \le 2\pi$, and $d - c \le \pi$

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\ d\rho d\theta d\phi$$

Note: Spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region.

Example: Evaluate $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

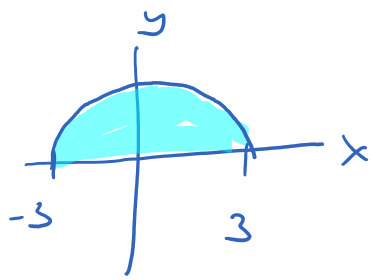
Top $z = \sqrt{9-x^2-y^2}$ ← Top part of a sphere of Radius 3

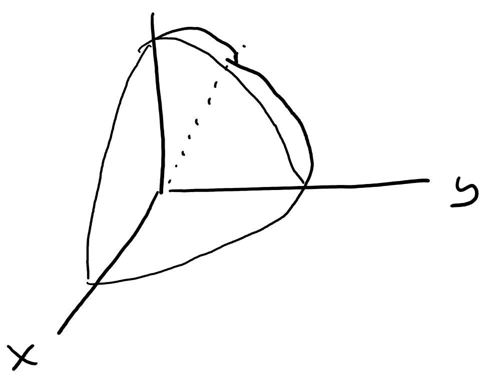
Bottom $z = 0$

Region D

$-3 \le x \le 3$

$0 \le y \le \sqrt{9-x^2}$





$0 \le \rho \le 3$

$0 \le \theta \le \pi$

$0 \le \phi \le \frac{\pi}{2}$

$f(x,y,z) = z^2\sqrt{x^2+y^2+z^2} = \rho^2\cos^2\phi\sqrt{\rho^2}$

$= \rho^3\cos^2\phi$

$$\int_0^{\pi/2}\int_0^{\pi}\int_0^{3} \rho^3\cos^2\phi\;\rho^2\sin\phi\;d\rho\,d\theta\,d\phi$$

$$\int_{\phi=0} \int_{\theta=0} \int_{\rho=0} \rho \cos^2\phi \;\rho \sin\phi \; d\rho \, d\theta \, d\phi$$

$$= \underbrace{\int_{\phi=0}^{\pi/2} \cos^2\phi \sin\phi \, d\phi}_{u = \cos\phi} \cdot \underbrace{\int_{\theta=0}^{\pi} 1 \, d\theta}_{} \cdot \int_{\rho=0}^{3} \rho^5 \, d\rho$$

$$= -\frac{1}{3}\cos^3\phi \Big|_0^{\pi/2} \cdot \pi \cdot \frac{1}{6}\rho^6 \Big|_0^3$$

$$= \left[-\frac{1}{3}\cos^3\left(\frac{\pi}{2}\right) - \frac{-1}{3}\cos^3(0)\right] \cdot \pi \cdot \frac{3^6}{6}$$

$$= \frac{1}{3} \cdot \pi \cdot \frac{3^6}{6} = \frac{81\pi}{2}$$

Example: Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

$\hookrightarrow \; x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$

$z^2 = x^2 + y^2$

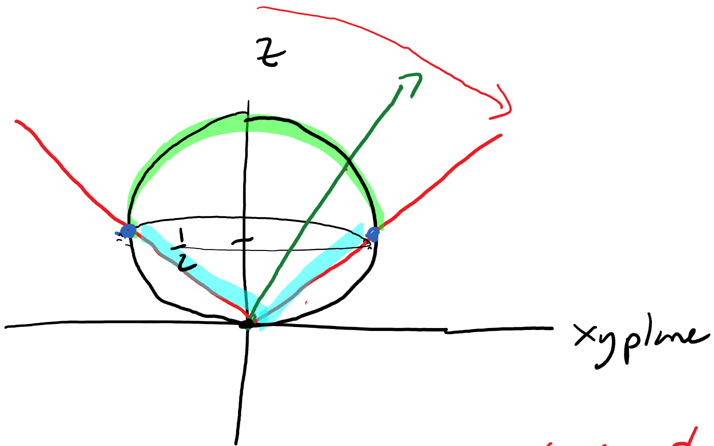
$\underbrace{x^2 + y^2} + z^2 = z$

$2z^2 = z$

$2z^2 - z = 0$

$z(2z - 1) = 0$

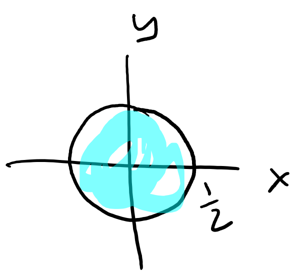
$z = 0 \quad z = \frac{1}{2}$



$0 \le \rho \le \cos\phi$

$\rho^2 = \rho\cos\phi$

$\rho = \cos\phi$



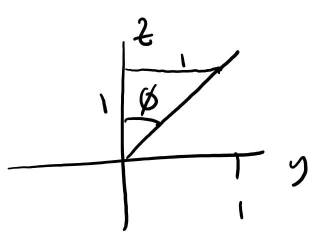
$0 \le \theta \le 2\pi$

so ϕ is the formula for the cone.

$z = \sqrt{x^2 + y^2}$

Let $x = 0 \qquad z = y$

$0 \le \phi \le \frac{\pi}{4}$



$\tan\phi = \frac{1}{1} = 1$

$\phi = \frac{\pi}{4}$

$$\iiint_E 1 \, dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} 1 \; \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta$$

$$\iiint_E \quad \int_{\theta=0} \int_{\phi=0} \int_{\rho=0} 1 \ \rho \sin\phi \ d\rho \ d\phi$$

$$= \int_{\theta=0}^{2\pi} 1 \, d\theta \cdot \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi$$

$$= 2\pi \cdot \int_{\phi=0}^{\pi/4} \frac{1}{3}\rho^3 \sin\phi \Big|_0^{\cos\phi} d\phi$$

$$= 2\pi \int_{\phi=0}^{\pi/4} \frac{1}{3}\cos^3\phi \sin\phi \ d\phi$$

$u = \cos\phi$

$$= 2\pi \cdot -\frac{1}{3} \cdot \frac{1}{4} \cos^4(\phi) \Big|_{\phi=0}^{\pi/4}$$

$$= -\frac{2\pi}{6}\cos^4\left(\frac{\pi}{4}\right) - -\frac{2\pi}{6}\cos^4(0)$$

$$= -\frac{2\pi}{6}\left(\frac{\sqrt{2}}{2}\right)^4 + \frac{2\pi}{6}$$

$-2\pi \quad 4 \qquad \pi$

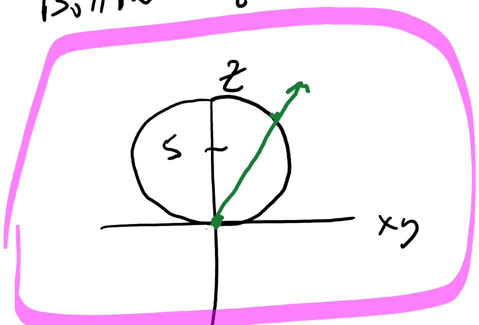
$$= \frac{-2\pi}{6} \cdot \frac{4}{16} + \frac{2\pi}{6} = \frac{\pi}{8}$$

Example: Convert the triple integral to spherical.

$$\int_{x=0}^{5}\int_{y=-\sqrt{25-x^2}}^{0}\int_{z=5-\sqrt{25-x^2-y^2}}^{5+\sqrt{25-x^2-y^2}} (x^2+y^2+z^2)^{1.5}\, dz dy dx$$

Top $z = 5 + \sqrt{25-x^2-y^2}$

Bottom $z = 5 - \sqrt{25-x^2-y^2}$



$0 \le \rho \le 10\cos\phi$

$z - 5 = \sqrt{25-x^2-y^2}$

$(z-5)^2 = 25-x^2-y^2$

$x^2+y^2+(z-5)^2 = 25$

$x^2+y^2+z^2-10z+25=25$

$x^2+y^2+z^2=10z$

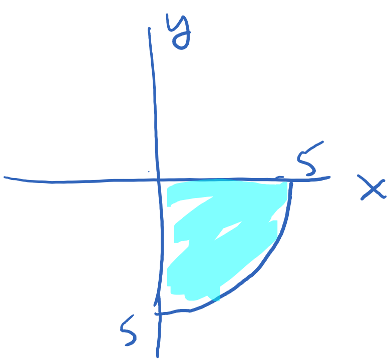
$\rho^2 = 10\rho\cos\phi$

$\rho = 10\cos\phi$

Region D

$0 \le x \le 5$ $\quad -\sqrt{25-x^2} \le y \le 0$

$0 \le \phi \le \frac{\pi}{2}$

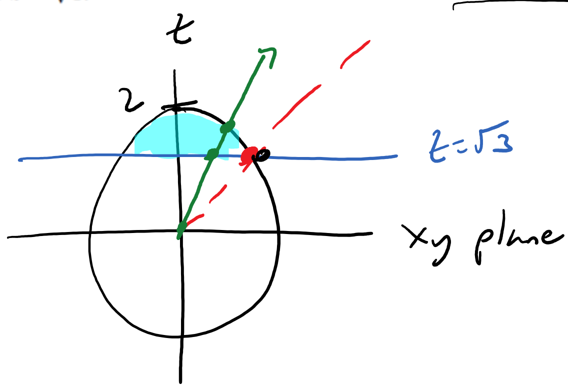


$\frac{3\pi}{2} \le \theta \le 2\pi$

$(\rho^2)^{1.5} = (\rho^2)^{3/2} = \rho^3$

$$\int_{\theta=\frac{3\pi}{2}}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{10\cos\phi} (\rho^2)^{1.5}\cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

Example Find the volume that is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the plane $z = \sqrt{3}$.



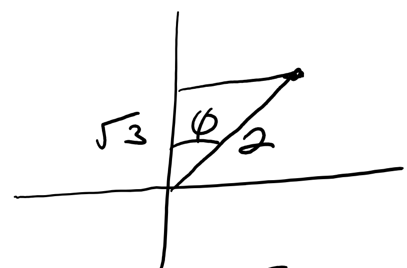
$0 \le \theta \le 2\pi$

$\sqrt{3}\sec\phi \le \rho \le 2$

$0 \le \phi \le \frac{\pi}{6}$

$\rho\cos\phi = \sqrt{3}$

$\rho = \sqrt{3}\sec\phi$



$\cos\phi = \frac{\sqrt{3}}{2}$

$\phi = \frac{\pi}{6}$

$$\int_{\phi=0}^{\pi/6}\int_{\theta=0}^{2\pi}\int_{\rho=\sqrt{3}\sec\phi}^{2} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_{\theta=0}^{2\pi} 1 \, d\theta \cdot \int_{\phi=0}^{\pi/6}\int_{\sqrt{3}\sec\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi$$

$\theta=0 \qquad \phi=0 \quad \rho=\sqrt{3}\sec\phi$

$$= 2\pi \cdot \int_{\phi=0}^{\pi/6} \frac{1}{3}\rho^3 \sin\phi \Big|_{\sqrt{3}\sec\phi}^{2} \, d\phi$$

$$= 2\pi \cdot \int_{\phi=0}^{\pi/6} \frac{8}{3}\sin\phi - \frac{3\sqrt{3}}{3}\sec^3\phi \sin\phi \, d\phi$$

$$= 2\pi \int_{\phi=0}^{\pi/6} \frac{8}{3}\sin\phi - \sqrt{3}\,\frac{\sin\phi}{\cos^3\phi} \, d\phi$$

$$= 2\pi \int_{\phi=0}^{\pi/6} \frac{8}{3}\sin\phi - \sqrt{3}\,\frac{\sin\phi}{\cos\phi}\,\frac{1}{\cos^2\phi} \, d\phi$$

$$= 2\pi \int_{\phi=0}^{\pi/6} \frac{8}{3}\sin\phi - \sqrt{3}\tan\phi\sec^2\phi \, d\phi$$

$$= 2\pi \left[\int^{\pi/6} \frac{8}{3}\sin\phi \, d\theta - \int^{\frac{\pi}{6}} \sqrt{3}\tan\phi\sec\phi\sec\phi \, d\phi \right]$$

$$= 2\pi \left[\int_{\phi=0} \frac{8}{3} \sin\phi \, d\theta - \int_{\theta=0} \sqrt{3} \tan\phi \sec\phi \sec\phi \, d\phi \right]$$

$u = \sec\phi$

$du = \sec\phi \tan\phi \, d\phi$

$$= 2\pi \left[\frac{8}{3} - \frac{3}{2}\sqrt{3} \right]$$