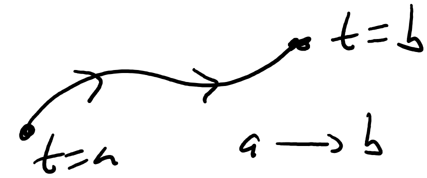
Section 16.2: Line Integrals

Reminder: In section 13.3 we discussed arc length of a space curve, $\mathbf{r}(t)$, on the interval $a \le t \le b$. The length of the curve, L is given by

$$L = \int_a^b ds = \int_a^b |\mathbf{r}'(t)|\ dt.$$

$$\boxed{ds = |r'(t)|\, dt}$$

Line integrals on a plane:
Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ or by the vector function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$.

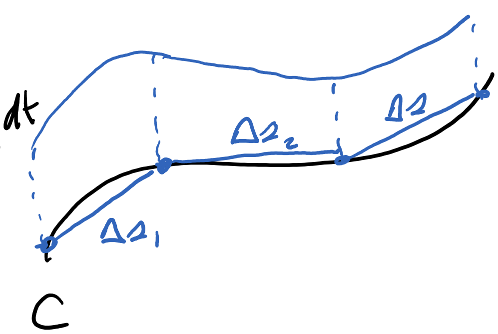


Definition: If f is defined on a smooth curve C, as defined above, then the **line integral of f along C** is

$$\int_C f(x,y)ds = \lim_{||P||\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \qquad \text{if the limit exists.}$$

If f is a continuous function, then we can compute this line integral by

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_a^b f(x(t),y(t))\,|r'(t)|\,dt$$

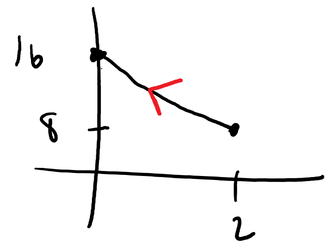


Note: This is sometimes refers to as the **line integral with respect to arc length**.

Note: The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

When $f(x,y) \ge 0$, the line integral of f along C represents the area of one side of the "fence" or "curtain" whose base is C and whose height at any point on the curve is $f(x,y)$. If $f(x,y) = 1$, then the line integral of f along C is the arc length of the curve C.

Example: Evaluate $\int_C (x^2 + y)\, ds$ where C consists of the line segment from the point $(2, 8)$ to $(0, 16)$



$$m = \frac{16-8}{0-2} = \frac{8}{-2} = -4$$

$$y = -4x + 16$$

$$\boxed{\begin{gathered} x = t \\ y = -4t + 16 \\ 0 \le t \le 2 \end{gathered}}$$

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = -4$$

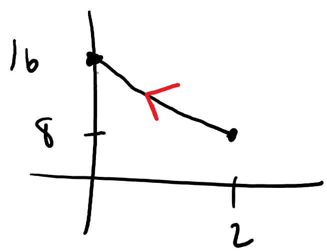
$$\int_0^2 \left[t^2 + (-4t+16)\right] \cdot \sqrt{(1)^2 + (-4)^2}\, dt$$

$$\sqrt{17} \int_0^2 t^2 - 4t + 16\, dt = \sqrt{17} \left[\frac{t^3}{3} - 2t^2 + 16t \right]_0^2$$

$$= \sqrt{17} \left(\frac{8}{3} - 8 + 32 \right) = \frac{80\sqrt{17}}{3}$$

Pg 2b

Example: Evaluate $\int_C (x^2 + y)\, ds$ where C consists of the line segment from the point $(2, 8)$ to $(0, 16)$



$m = \frac{16-8}{0-2} = \frac{8}{-2} = -4$

$y = -4x + 16$

$4x = 16 - y$

$x = 4 - \frac{y}{4}$

$x = 4 - \frac{t}{4}$ $\qquad y = t$

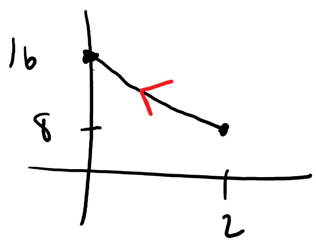
$8 \le t \le 16$

$\frac{dx}{dt} = -\frac{1}{4}$ $\qquad \frac{dy}{dt} = 1$

$$\int_8^{16} \left(\left(4 - \tfrac{1}{4}y\right)^2 + t \right) \cdot \sqrt{\left(-\tfrac{1}{4}\right)^2 + (1)^2}\, dt = \cdots = \frac{80\sqrt{17}}{3}$$

Example: Evaluate $\int_C (x^2+y)\, ds$ where C consists of the line segment from the point $(2,8)$ to $(0,16)$

Shortcut for a line segment



$r(t) = (1-t)\,(\text{start point}) + t\,(\text{end point})$

$r(t) = (1-t)\langle 2,8\rangle + t\langle 0,16\rangle$

$= \langle 2-2t,\ 8-8t\rangle + \langle 0,16t\rangle$

$r(t) = \langle \underbrace{2-2t}_{x},\ \underbrace{8+8t}_{y}\rangle$

$0 \le t \le 1$

$\frac{dx}{dt} = -2$

$\frac{dy}{dt} = 8$

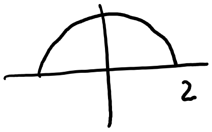
$$\int_C (x^2+y)\,ds = \int_0^1 \left((2-2t)^2 + (8+8t)\right)\sqrt{(-2)^2+(8)^2}\,dt$$

$$= \sqrt{68}\int_0^1 (4-8t+4t^2+8+8t)\,dt$$

$$= \sqrt{68}\int_0^1 (12+4t^2)\,dt = \sqrt{68}\left(12t+\frac{4t^3}{3}\right)\Big|_0^1$$

$$= \sqrt{68}\left(12+\frac{4}{3}\right) = \sqrt{68}\cdot\frac{40}{3}$$

Example: Evaluate $\int_C (2 + x^2 y)\,ds$, where C is the upper half of the circle $x^2 + y^2 = 4$.



$y = \sqrt{4 - x^2}$

$$\boxed{\begin{array}{ll} x = t & y = \sqrt{4 - t^2} \\ -2 \le t \le 2 & \end{array}}$$

$X = x \qquad y = \sqrt{4 - x^2}$

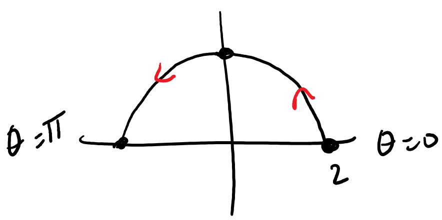
$-2 \le x \le 2$

$x' = dx = 1$

$y' = dy = \frac{1}{2}(4 - x^2)^{-1/2} \cdot (-2x)$

$= \dfrac{-x}{\sqrt{4 - x^2}}$

$$\int_{-2}^{2} \left(2 + x^2 \sqrt{4 - x^2}\right) \sqrt{(1)^2 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2}\; dx$$



$x = 2\cos\theta \qquad 0 \le \theta \le \pi$

$y = 2\sin\theta$

$\dfrac{dx}{d\theta} = -2\sin\theta$

$\dfrac{dy}{d\theta} = 2\cos\theta$

$$\int_C (2 + x^2 y)\,ds = \int_0^{\pi} \left(2 + 4\cos^2\theta \cdot 2\sin\theta\right) \sqrt{(-2\sin\theta)^2 + (2\cos\theta)^2}\; d\theta$$

$$= \int 2 + 8\cos^2\theta \sin\theta)\sqrt{4\sin^2\theta + 4\cos^2\theta}\; d\theta$$

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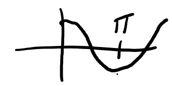
$$= \int_0^\pi \left(2 + 8\cos^2\theta\sin\theta\right)\sqrt{4}\, d\theta$$

$$= \int_0^\pi 4 + 16\cos^2\theta\sin\theta \, d\theta = \int_0^\pi 4\, d\theta + \underbrace{\int_0^\pi 16\cos^2\theta\sin\theta\, d\theta}_{u = \cos\theta}$$

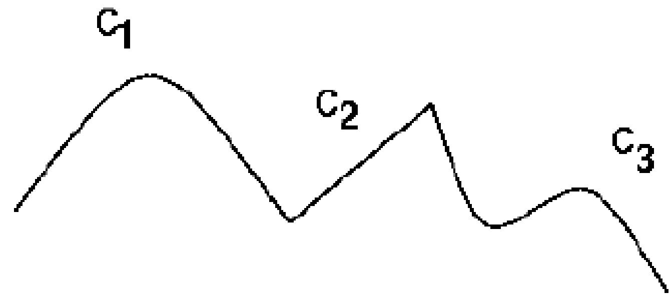
$$= \left(4\theta - \frac{16}{3}\cos^3\theta\right)\Big|_0^\pi$$



$$= 4\pi - \frac{16}{3}(\cos\pi)^3 - \left(0 - \frac{16}{3}(1)^3\right)$$

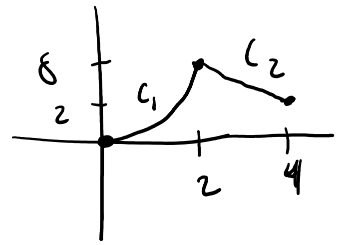
$$= 4\pi + \frac{16}{3} + \frac{16}{3} = 4\pi + \frac{32}{3}$$

Definition: If C is a **piecewise-smooth curve**, that is C is made up a collection of smooth curves where one curve ends then the next curve begins, then line integral of f along C is defined to be the sum of the integrals of f along each smooth piece of C.



$$\int_C f(x,y)ds = \underline{\int_{C_1} f(x,y)ds} + \underline{\int_{C_2} f(x,y)ds} + \underline{\int_{C_3} f(x,y)ds}$$

Example: Evaluate $\int_C 2y\,ds$, where C consists of C_1 of $y = x^3$ from $(0,0)$ to $(2,8)$ followed by the line segment from $(2,8)$ to $(4,2)$.



C_1) $\quad y = x^3 \qquad x = t \quad y = t^3 \quad 0 \le t \le 2$

$$\int_{C_1} 2y\,ds = \int_0^2 2t^3 \cdot \sqrt{(1)^2 + (3t^2)^2}\;dt$$

$$= \int_0^2 2t^3\sqrt{1+9t^4}\;dt$$

$u = 1+9t^4$

$du = 36t^3\,dt$

$\frac{1}{36}\,du = t^3\,dt$

$$= \int_1^{145} \frac{2}{36}\sqrt{u}\;du = \frac{1}{18}\,u^{3/2}\cdot\frac{2}{3}\Big|_1^{145}$$

$$= \frac{1}{27}\left(145^{3/2} - 1\right)$$

C_2)

$$r(t) = (1-t)\langle 2,8\rangle + t\langle 4,2\rangle$$
$$= \langle 2-2t,\ 8-8t\rangle + \langle 4t,\ 2t\rangle$$
$$= \langle 2+2t,\ 8-6t\rangle \qquad 0 \le t \le 1$$

$$\int_{C_2} 2y\,ds = \int_0^1 2(8-6t)\sqrt{(2)^2+(-6)^2}\;dt$$

$$\int_{C_2} \cdots \qquad \int_0 \cdots$$

$$= \int_0^1 (16 - 12t)\sqrt{40}\, dt$$

$$= \sqrt{40}\left[16t - 6t^2\right]_0^1 = \sqrt{40}\,(16 - 6) = 10\sqrt{40}$$

$$= 20\sqrt{10}$$

Answer: $\frac{1}{27}\left(145^{3/2} - 1\right) + 20\sqrt{10}$

Pg 6: line integrals with respect to x and to y

$$ds = \sqrt{(x')^2 + (y')^2}\, dt$$

$$dx = x'(t)\, dt$$

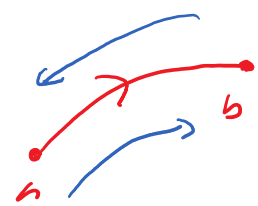
Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ for $a \le t \le b$.

The **line integral of f along C with respect to x** is $\int_C f(x,y)dx = \int_a^b f(x(t),y(t))\, x'(t)\, dt$

The **line integral of f along C with respect to y** is $\int_C f(x,y)dy = \int_a^b f(x(t),y(t))\, y'(t)\, dt$

Note: **For these integrals, the orientation of the curve, which direction is is traversed, is important.** If C and $-C$ represent traversing the same curve but in different directions, then

$$\int_{-C} f(x,y)dx = -\int_C f(x,y)dx$$



Example: Evaluate $\int_C y^2dx + xdy$, where C is the line segment from $(-5,-3)$ to $(3,1)$.

$$\int_C y^2\,dx + \int_C x\,dy$$

$$r(t) = (1-t)\langle -5,-3\rangle + t\langle 3,1\rangle$$

$$= \langle -5+5t, -3+3t\rangle + \langle 3t, t\rangle$$

$$= \langle -5+8t, -3+4t\rangle \qquad 0 \le t \le 1$$

$$\int_0^1 (-3+4t)^2 \cdot 8 + (-5+8t)4 \quad dt$$

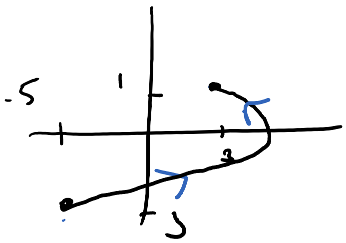
$$\int_0^1 (9-24t+16t^2)8 - 20 + 32t \quad dt$$

$$\int_0^1 72 - 192t + 128t^2 - 20 + 32t \quad dt$$

$$\int_0^1 52 - 160t + 128t^2 \, dt = \left(52t - 80t^2 + \frac{128}{3}t^3\right)\Big|_0^1$$

$$= 52 - 80 + \frac{128}{3} = \frac{44}{3}$$

Example: Evaluate $\int_C y^2 dx + x dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.



$y = t \qquad x = 4 - t^2$

$-3 \le t \le 1$

$$\int_{-3}^{1} t^2(-2t) + (4-t^2)1 \; dt = \int_{-3}^{1} -2t^3 + 4 - t^2 \; dt$$

$$= \left(-\frac{2t^4}{4} + 4t - \frac{t^3}{3}\right)\Bigg|_{-3}^{1}$$

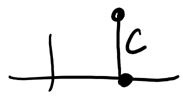
$$= \left(-\frac{2}{4} + 4 - \frac{1}{3}\right) - \left(\frac{-2(81)}{4} - 12 + 9\right)$$

$$= \frac{14}{3}$$

Example: Consider $f(x,y) = x^2$ and C be the smooth curve $r(t) = \langle t, 0 \rangle$ for $0 \le t \le 2$. Then $x(t) = t$ and $y(t) = 0$ and $x'(t) = 1\ dt$

$$\int_C f(x,y)dx = \int_0^2 t^2 * 1\ dt = \int_0^2 t^2 dt$$

Compare this to integrating $y = x^2$ on the interval $[0,2]$ which is $\int_0^2 x^2 dx$



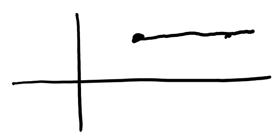
$$r(t) = (1-t)\langle 3,0\rangle + t\langle 3,15\rangle$$
$$= \langle 3-3t, 0\rangle + \langle 3t, 15t\rangle$$
$$= \langle \underline{\underline{3}}, 15t\rangle$$

Example: The curve C is the line segment from $(3,0)$ to $(3,15)$.

Compute $\int_C (x^2 + 2y)dx$

$$= \int_0^1 \left((3)^2 + 2(15t)\right) 0\ dt = \int_0^1 0\ dt$$

$$= 0$$



Example: The curve C is the line segment from $(2,5)$ to $(7,5)$.

Compute $\int_C (x^2 + 2y)\underline{\underline{dy}} = 0$

Line Integrals in Space:

Let C be a smooth curve defined by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$. The line integral of f along C is

$$\int_C f(x,y,z)ds = \int_a^b f(x(t), y(t), z(t))\ |\mathbf{r}'(t)|\ dt$$

Note: The line integral of f with respect to x, with respect to y, and with respect to z are defined in a manner similar to before.

$$ds = |r'(t)|\ dt$$
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

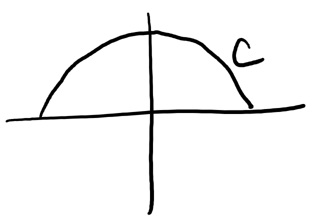
Physical interpretation of a line integral: Let $\rho(x,y)$ represent the linear density at a point (x,y) of a thin wire shaped like the curve C.

Then the mass of the wire is $m = \int_C \rho(x,y)ds$

The center of mass $(\overline{x}, \overline{y})$ is

$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)ds$ and $\overline{y} = \frac{1}{m}\int_C y\rho(x,y)ds$

Example: A thin wire with linear density $\rho(x,y) = 2 + x^2y$ takes the shape of the semicircle $x^2 + y^2 = 4$, $y \geq 0$. Find the mass of this wire.

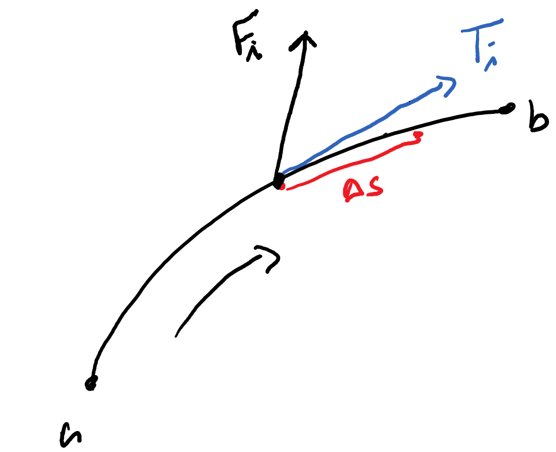


$$\int_C 2 + x^2y \, ds = \cdots = 4\pi + \frac{32}{3}$$

Pg 12: line integrals in vector fields

Line Integrals of Vector Fields

Suppose $\mathbf{F}$ is a continuous vector field (i.e. force field). Find the work done moving a particle along the curve C given by $\mathbf{r}(t)$ for $a \le t \le b$.



$$W = F \cdot \underline{d}$$

$$d_i \approx T_i \Delta s$$

$$W \approx \sum F_i \cdot T_i \Delta s$$

$$W = \int_a^b F \cdot T \, ds = \int_C F \cdot T \, ds$$

$$T = \frac{r'(t)}{|r'(t)|}$$

$$= \int_C F \cdot \frac{r'(t)}{|r'(t)|} \, ds$$

$$= \int_C F \cdot \frac{r'(t)}{|r'(t)|} \, |r'(t)| \, dt$$

$$W = \int_C F \cdot r'(t) \, dt$$

Definition: Let **F** be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of F along C** is

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_c \mathbf{F} \cdot \mathbf{T} \, ds$$

$dr = r'(t) \, dt$

Note: If the orientation of the curve is changed, i.e. C is replaced by $-C$, then the unit tangent vector is replaced by its negative. Thus

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

Example: Find the work done by the force field **F** in moving a particle along the curve C.

$\mathbf{F}(x, y, z) = \langle xy, yz, xz \rangle$

C: $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$.

$$F(r(t)) = \langle t \cdot t^2, t^2 \cdot t^3, t \cdot t^3 \rangle = \langle t^3, t^5, t^4 \rangle$$

$$r'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$W = \int_C F \cdot dr = \int_C F \cdot r'(t)\, dt$$

$$= \int_0^1 t^3 + 2t^6 + 3t^6\, dt = \int_0^1 t^3 + 5t^6\, dt$$

$$= \frac{t^4}{4} + \frac{5t^7}{7} \Big|_0^1 = \frac{1}{4} + \frac{5}{7} = \frac{27}{28}$$

Relationship between a line integral over a vector field and line integrals with respect to x, y, and z.

$F = \langle P, Q, R \rangle$

Let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ with

C defined by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle = \int_C P dx + Q dy + R dz$$