Section 16.3: The Fundamental Theorem for Line Integrals

Recall the Fundamental Theorem of Calculus: $\int_a^b F'(x)dx = F(b) - F(a)$

$F = \nabla f = \langle f_x, f_y, f_z \rangle$

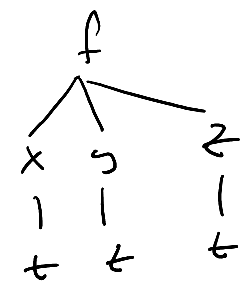
$r(t) = \langle x(t), y(t), z(t) \rangle$

$r'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

Theorem: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \int_C \nabla f \cdot r'(t)\, dt$$

$$= \int_a^b f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \quad dt$$



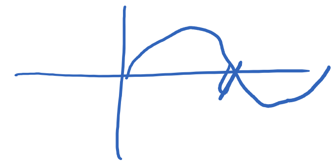
$$= \int_a^b \frac{d}{dt} f(r(t)) \quad dt$$

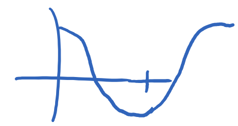
$$= f(r(t)) \Big|_a^b = f(r(b)) - f(r(a))$$

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

Note: The line integral of a conservative vector field (∇f with potential function f) can be evaluated by knowing the endpoints of the curve.

Note: This can also be used on curves that are that are piecewise smooth.





Example: Let $f(x, y) = 3x + x^2y - y^3$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \nabla f$ and C is the curve given by $\mathbf{r}(t) = \langle e^t \sin(t), e^t \cos(t) \rangle$, $0 \leq t \leq \pi$.

$$r(0) = \langle e^0 \sin(0), e^0 \cos(0) \rangle = \langle 0, 1 \rangle$$

$$r(\pi) = \langle e^\pi \sin(\pi), e^\pi \cos(\pi) \rangle = \langle 0, -e^\pi \rangle$$

$$\int_C F \cdot dr = f(0, -e^\pi) - f(0, 1)$$

$$= 0 + 0 - (-e^\pi)^3 - \left(0 + 0 - (1)^3\right)$$

$$= \underline{e^{3\pi} + 1}$$

Definition: If $\mathbf{F}$ is a continuous vector field with domain D, we say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if and only if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 with the same starting and ending points.

Note: **Line integrals of conservative vectors fields are independent of path.**

Theorem: $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D. (A closed path is one that starts and stops at the same point.)

An interpretation is that the work done by a conservative vector field as an object moves around a closed path is 0.

Question: How do we determine if a vector field is conservative and if so, can we find the potential function?

Theorem: IF $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then we have $P_y = Q_x$.

$f_x = P \qquad f_y = Q$

$F = \nabla f = \langle f_x, f_y \rangle$

$P_y = f_{xy} = f_{yx} = Q_x$

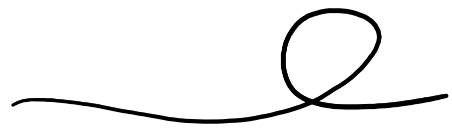
Example: Is $\mathbf{F} = \langle \underbrace{3x^2 - 4y}_{P}, \underbrace{4y^2 - 2x}_{Q} \rangle$ a conservative vector field?

$$\underbrace{P_y = -4 \qquad Q_x = -2}_{\text{not equal}}$$

so F is not conservative

Definition: A **simple curve** is a curve that does not intersect itself anywhere between its endpoints. A **simply-connected region** in the plane is one that is connected and does not have holes. i.e. every simple closed curve encloses only points in the region.

Simple



not simple.

Theorem: Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field on an **open simply-connected region** D. Suppose that P and Q have continuous first-order derivatives and $P_y = Q_x$ throughout D. Then $\mathbf{F}$ is conservative.

Note: The above criteria to determine if a vector field is conservative works only for $\Re^2$. The criteria for a vector field in $\Re^3$ is found in section 16.5.

$P \quad Q$

Example: Determine whether $\mathbf{F} = \langle x + y^2, 2xy + y^2 \rangle$ is conservative or not. If so, find a potential function.

$P_y = 2y$
$Q_x = 2y$ equal so F is conservative

$$P = f_x = x + y^2 \longrightarrow f(x,y) = \frac{x^2}{2} + xy^2 + C(y)$$

$$Q = f_y = 2xy + y^2 \longrightarrow f(x,y) = xy^2 + \frac{y^3}{3} + C(x)$$

$$\boxed{f(x,y) = xy^2 + \frac{1}{2}x^2 + \frac{1}{3}y^3}$$

Example: Given that $\mathbf{F} = \langle 4xe^z, \cos(y), 2x^2e^z \rangle$ is conservative. Compute $\int_C \mathbf{F} \cdot dr$ where $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$, for $0 \leq t \leq \frac{\pi}{2}$.

$r(0) = \langle 0, 0, 1 \rangle$ $\quad r\left(\frac{\pi}{2}\right) = \langle 1, \frac{\pi}{2}, 0 \rangle$

$$f_x = 4xe^z \longrightarrow f(x,y,z) = 2x^2e^z + c(y,z)$$

$$f_y = \cos(y) \longrightarrow f(x,y,z) = \sin(y) + c(x,z)$$

$$f_z = 2x^2e^z \longrightarrow f(x,y,z) = 2x^2e^z + c(x,y)$$

$$f(x,y,z) = 2x^2e^z + \sin(y)$$

$(0, 0, 1)$ $\quad (1, \frac{\pi}{2}, 0)$

$$\int_C F \cdot dr = f\left(1, \frac{\pi}{2}, 0\right) - f(0, 0, 1)$$

$$= 2 + 1 - (0 + 0) = 3$$

Example: Given $\mathbf{F} = \langle 2xy^3 + z^2, 3x^2y^2 + 2yz, y^2 + 2xz \rangle$ is conservative. Find a potential function F.

$$f_x = 2xy^3 + z^2 \longrightarrow f(x,y,z) = x^2y^3 + xz^2 + C(y,z)$$

$$f_y = 3x^2y^2 + 2yz \longrightarrow f(x,y,z) = x^2y^3 + y^2z + C(x,z)$$

$$f_z = y^2 + 2xz \longrightarrow f(x,y,z) = y^2z + xz^2 + C(x,y)$$

$$f(x,y,z) = x^2y^3 + xz^2 + y^2z$$