A space curve is parametrized by the vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

A surface, $z = f(x, y)$, is parametrized by a vector function of two variables.
$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ with (u, v) in region D.

Useful Parametrizations

- Surface given by $z = f(x, y)$.

Example: $z = x^2 + 4y^2$

$x = u$
$y = v$
$z = u^2 + 4v^2$
$r(u,v) = \langle u, v, u^2+4v^2\rangle$

$x = x$
$y = y$
$z = x^2 + 4y^2$
$r(x,y) = \langle x, y, x^2+4y^2\rangle$

Example: $x = z^2 + y^2$

$x = z^2 + y^2$
$y = y$
$z = z$

$r(y,z) = \langle z^2+y^2, y, z\rangle$

- Surface in cylindrical coordinates

Example: $x^2 + y^2 = 9$ for $0 \leq z \leq 2$

$x = 3\cos\theta$

$y = 3\sin\theta$

$z = z$

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq 2$

$r(z, \theta) = \langle 3\cos\theta, 3\sin\theta, z \rangle$

- Surface in spherical coordinates

Example: $x^2 + y^2 + z^2 = 4$

$x = 2 \sin\phi \cos\theta$

$y = 2 \sin\phi \sin\theta$

$z = 2 \cos\phi$

$r(\phi, \theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

Example: Find the tangent plane to the surface with parametric equations given below at the point $(1, 4, 5)$.

$\mathbf{r}(u, v) = \langle u^3, v^2, u + 2v \rangle$

$x = u^3$ $y = v^2$

$1 = u^3$ $4 = v^2$

$u = 1$ $v = \pm 2$

$u = 1 \quad v = 2$

$r(1,2) = \langle 1, 4, 5 \rangle$ ✓

$u = 1 \quad v = -2$

✗ $r(1,-2) = \langle 1, 4, -3 \rangle$

$r_u = \langle 3u^2, 0, 1 \rangle$

$r_u(1,2) = \langle 3, 0, 1 \rangle$

$r_v = \langle 0, 2v, 2 \rangle$

$r_v(1,2) = \langle 0, 4, 2 \rangle$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 3 & 0 & 1 \\ 0 & 4 & 2 \end{vmatrix}$$

$$= (0-4)i - (6-0)j + (12-0)k$$

$$= \langle -4, -6, 12 \rangle$$

Tangent plane $\boxed{-4(x-1) - 6(y-4) + 12(z-5) = 0}$

Note: In the special case the surface is defined by $z = f(x, y)$ and is parametrized by
$\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$. Then a normal vector is

$r_x = \langle 1, 0, f_x \rangle$

$r_y = \langle 0, 1, f_y \rangle$

normal vector
$r_x \times r_y = \langle -f_x, -f_y, 1 \rangle$

$\langle f_x, f_y, -1 \rangle$

Example: Find a normal vector for the surface defined as $x = f(y, z)$

$r(y, z) = \langle f(y, z), y, z \rangle$

Cross product
$\langle 1, -f_y, -f_z \rangle$

Definition: If a smooth parametric surface S is given by the equation $\mathbf{r}(u, v)$ and S is covered just once as (u, v) ranges throughout the parametric domain D, then the **surface area** of S is

$$A(S) = \iint_D dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

$$z = f(x, y)$$

$$\iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1} \; dA$$

Example: Find the surface area for the surface given by $x = uv$, $y = u + v$, and $z = u - v$ where $u^2 + v^2 \leq 1$

$$r(u,v) = \langle uv, u+v, u-v \rangle$$

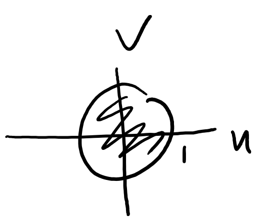
$$r_u = \langle v, 1, 1 \rangle$$

$$r_v = \langle u, 1, -1 \rangle$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix}$$

$$= (-1-1)i - (-v-u)j + (v-u)k$$

$$= \langle -2, v+u, v-u \rangle$$



$$A(S) = \iint_D \sqrt{4 + (v+u)^2 + (v-u)^2} \; dA$$

$$= \iint_D \sqrt{4 + v^2 + 2uv + u^2 + v^2 - 2uv + u^2} \; dA$$

$$= \iint_D \sqrt{4 + 2v^2 + 2u^2} \; dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\sqrt{4+2r^2} \; dr\,d\theta$$

$$= \int^{2\pi} \int^{1}$$

$$= \int_0^{2\pi} 1\, d\theta \cdot \underbrace{\int_{r=0}^{1} r\sqrt{4+2r^2}\, dr}$$

$A = 4 + 2r^2$

$dA = 4r\,dr$

$\frac{1}{4} = r\,dr$

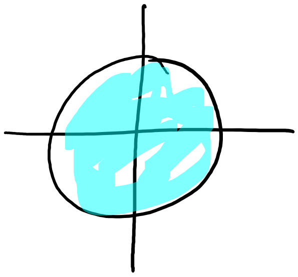
$$= 2\pi \int_4^6 \frac{1}{4}\sqrt{A}\ dA$$

$$= \frac{2\pi}{4} \frac{2}{3} \cdot A^{3/2} \Big|_4^6 = \frac{\pi}{3}\left(6^{3/2} - 4^{3/2}\right)$$

$$= 2\pi\left(\sqrt{6} - \frac{4}{3}\right)$$

$z = 8 - 2x - 2y$

Example: Find the surface area for the part of the plane $\underbrace{2x + 2y + z = 8}$ inside the cylinder $x^2 + y^2 = 9$.



$x = r\cos\theta$

$y = r\sin\theta$

$z = 8 - 2r\cos\theta - 2r\sin\theta$

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$r(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 8 - 2r\cos\theta - 2r\sin\theta \rangle$

$r_r = \langle \cos\theta,\ \sin\theta,\ -2\cos\theta - 2\sin\theta \rangle$

$r_\theta = \langle -r\sin\theta,\ r\cos\theta,\ 2r\sin\theta - 2r\cos\theta \rangle$

$r_r \times r_\theta = \langle 2r,\ 2r,\ r \rangle$

$|r_r \times r_\theta| = \sqrt{4r^2 + 4r^2 + r^2} = \sqrt{9r^2} = 3r$

$$A(S) = \iint_D |r_r \times r_\theta|\, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} 3r \, dr\, d\theta$$

$$= \int_0^{2\pi} 1\, d\theta \cdot \int_0^3 3r\, dr = 2\pi \cdot \frac{3r^2}{2}\Big|_0^3$$

$$= 2\pi \cdot \frac{27}{2} = \underline{27\pi}$$

Example: Find the surface area for the part of the plane $2x+2y+z=8$ inside the cylinder $x^2+y^2=9$.

$\langle -f_x, -f_y, 1 \rangle$

$x = x$

$y = y$

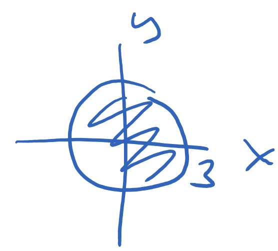
$z = 8 - 2x - 2y$

$r_x \times r_y = \langle -(-2), -(-2), 1 \rangle$

$= \langle 2, 2, 1 \rangle$

$|r_x \times r_y| = \sqrt{4+4+1} = \sqrt{9} = 3$

$$A(S) = \iint_D |r_x \times r_y| \, dA = \underbrace{\iint_D 3 \, dA}$$



$3 \iint_D dA = 3(\text{Area of Region } D)$

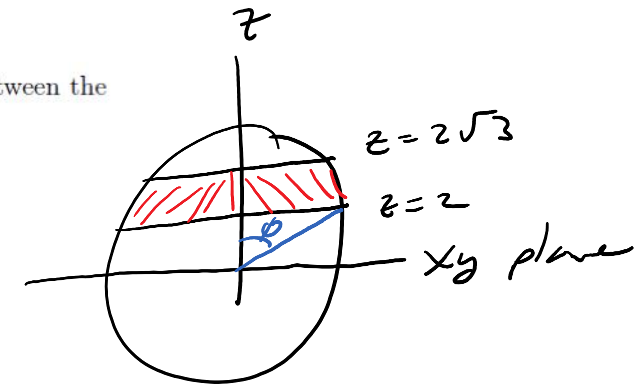
$= 3\pi(3)^2$

$= 27\pi$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{3} 3r \, dr \, d\theta$$

$= 27\pi$

Pg 8

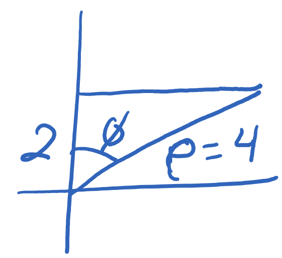


Example: Find the surface area of the sphere $x^2+y^2+z^2=16$ between the planes $z=2$ and $z=2\sqrt{3}$.

$x = 4\sin\phi\cos\theta$
$y = 4\sin\phi\sin\theta$
$z = 4\cos\phi$

where $0 \le \theta \le 2\pi$ and $\frac{\pi}{6} \le \phi \le \frac{\pi}{3}$

$$r_\phi \times r_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4\cos\phi\cos\theta & 4\cos\phi\sin\theta & -4\sin\phi \\ -4\sin\phi\sin\theta & 4\sin\phi\cos\theta & 0 \end{vmatrix}$$



$$r_\phi \times r_\theta = \langle 16\sin^2\phi\cos\theta,\ -16\sin^2\phi\sin\theta,\ 16\sin\phi\cos\phi\cos^2\theta + 16\sin\phi\cos\phi\sin^2\theta\rangle$$

$$r_\phi \times r_\theta = \langle 16\sin^2\phi\cos\theta,\ -16\sin^2\phi\sin\theta,\ 16\sin\phi\cos\phi\rangle$$

$$|r_\phi \times r_\theta| = \sqrt{16^2\sin^4\phi\cos^2\theta + 16^2\sin^4\phi\sin^2\theta + 16^2\sin^2\phi\cos^2\phi}$$

$$|r_\phi \times r_\theta| = \sqrt{16^2\sin^4\phi + 16^2\sin^2\phi\cos^2\phi}$$

$$|r_\phi \times r_\theta| = \sqrt{16^2\sin^2\phi(\sin^2\phi+\cos^2\phi)} = \sqrt{16^2\sin^2\phi} = 16\sin\phi$$
Note: $\sin\phi > 0$ on the given interval of ϕ.

$$S = \int_{\theta=0}^{2\pi}\int_{\phi=\pi/6}^{\pi/3} |r_\phi \times r_\theta|\, d\phi\, d\theta = \int_{\theta=0}^{2\pi}\int_{\phi=\pi/6}^{\pi/3} 16\sin\phi\ \, d\phi\, d\theta = \cdots = 16\pi(\sqrt{3}-1)$$