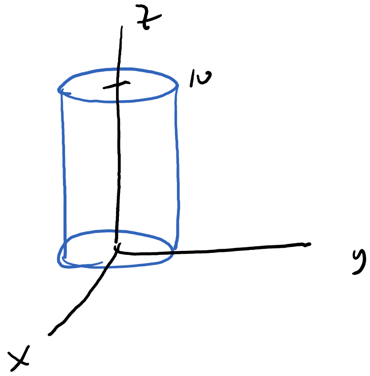
Section 16.9: Divergence Theorem

Divergence Theorem Let E be a simple solid region whose boundary surface has positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} dV$$

Example: Let E be the solid bounded by $x^2 + y^2 = 25$, $z = 0$, and $z = 10$. Find the flux of the vector field $\mathbf{F} = \langle 1 + x, 2 + 3y, 2z + 5 \rangle$ over the boundary of the solid. Use positive orientation.

$$\operatorname{div} F = P_x + Q_y + R_z = 1 + 3 + 2 = 6$$

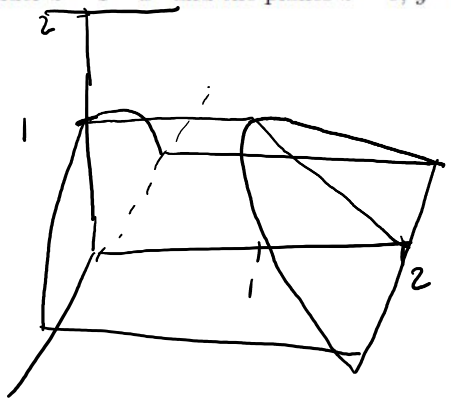


$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$$

$$= \iiint_E 6 \, dV = 6 \iiint_E 1 \, dV$$

$$= 6 \cdot \pi (5)^2 (10) = 1500\pi$$

Example: Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = \left\langle xy, y^2 + e^{xz^2}, \sin(xy)\right\rangle$ and S is the surface (with positive orientation) of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.

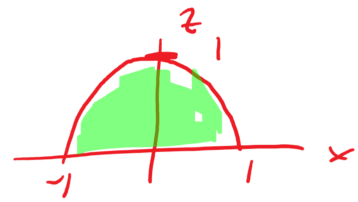


$\operatorname{div} F = y + 2y + 0 = 3y$

proj on xz plane

Left : $y = 0$

Right : $y = 2 - z$



$-1 \le x \le 1$

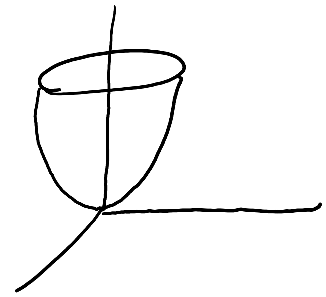
$0 \le z \le 1 - x^2$

$$\iint_S F\cdot dS = \iiint_E 3y\, dV$$

$$= \int_{x=-1}^{1}\int_{z=0}^{1-x^2}\int_{y=0}^{2-z} 3y\, dy\, dz\, dx = \cdots = \frac{184}{5}$$

Example: Let S be the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$. Let $\mathbf{F} = \langle xz, yz, 3z^2 \rangle$. Use positive orientation.

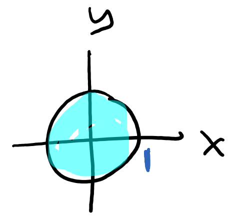
Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where S is the boundary of the solid.



$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$$

$$\operatorname{div} F = z + z + 6z = 8z$$

$$= \iiint_E 8z \, dV$$



Top $z = 1$

Bottom $z = x^2 + y^2 = r^2$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{r^2}^{1} 8z \, r \, dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 4z^2 r \Big|_{z=r^2}^{1} dr \, d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (4r - 4r^5) \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(2r^2 - \frac{4r^6}{6}\right)\Big|_0^1 d\theta = \int_{\theta=0}^{2\pi} 2 - \frac{4}{6} \, d\theta$$

$$= \int_0^{2\pi} \frac{4}{3} \, d\theta = \frac{4}{3} 2\pi = \frac{8\pi}{3}$$

Example: Let S_1 be the surface of the paraboloid $z = x^2 + y^2$ for $0 \le z \le 1$ with downward orientation. Let $\mathbf{F} = \langle xz, yz, 3z^2 \rangle$.

Compute $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}_1$

S_2 disk at $z=1$, $x^2+y^2=1$

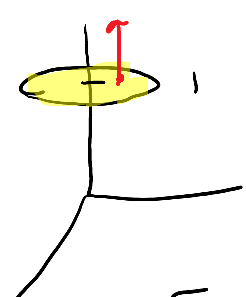
Let $S = S_1 \cup S_2$

$$\iint_S F \cdot dS = \iint_{S_1} F \cdot dS_1 + \iint_{S_2} F \cdot dS_2$$

$$\iint_{S_1} F \cdot dS = \iint_S F \cdot dS - \iint_{S_2} F \cdot dS_2$$

$$= \frac{8\pi}{3} - 3\pi = \boxed{-\frac{\pi}{3}}$$

S_2



$x = x$
$y = y$
$z = 1$

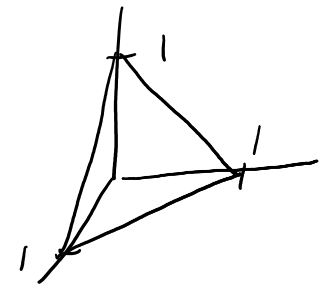
cross product
$\langle 0, 0, 1 \rangle$ Right direction

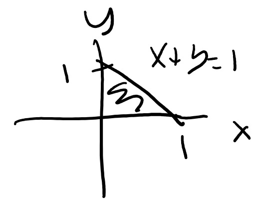
$$F = \langle xy, yz, 3z^2 \rangle = \langle xy, y, 3 \rangle$$

$$\iint F \cdot dS_2 = \iint F \cdot \langle 0, 0, 1 \rangle \, dA = \iint 3 \, dA$$

$$\iint_{S_2} F \cdot dS_2 = \iint_D F \cdot \langle 0,0,1\rangle \, dA = \iint_D 3 \, dA$$

$$= 3\pi(1)^2 = 3\pi$$





Example: Let S be the closed surface of a Tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F} = \langle y, z-y, x \rangle$. and use positive orientation.
Evaluate $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$

$$\operatorname{div} F = 0 + (-1) + 0$$

$$\iint\limits_S F \cdot dS = \iiint\limits_E \operatorname{div} F \, dV$$

$$= \iiint\limits_E -1 \, dV = \iint\limits_D \int_{z=0}^{1-x-y} (-1) \, dz \, dA$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} -1 \, dz \, dy \, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} -z \Big|_0^{1-x-y} dy \, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} -1 + x + y \quad dy \, dx$$

$$= \int_{x=0}^{1} -y + xy + \frac{y^2}{2} \Big|_0^{1-x} dx$$

$x=0$

$$= \int_{x=0}^{1} -1+x+x-x^2+\frac{1}{2}(1-x)^2 \, dx$$

$$= \int_{x=0}^{1} -1+2x-x^2+\frac{1}{2}\left(1-2x+x^2\right) \, dx$$

$\frac{1}{2} - x + \frac{1}{2}x^2$

$$= \int_{x=0}^{1} -\frac{1}{2}+x-\frac{1}{2}x^2 \, dx = -\frac{1}{2}x + \frac{x^2}{2} - \frac{1}{6}x^3 \Big|_0^1$$

$$= -\frac{1}{2} + \frac{1}{2} - \frac{1}{6} = -\frac{1}{6}$$