Math 251 | Exam 1 A | Thursday, February 8, 2024

Printed Name: ______________________

Section: Key

UIN: ______________________

Signature: ______________________

On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work.

Section 507: class time TR 12:45
Section 508: class time TR 2:20

- **Show all appropriate work to receive full credit.**
- **You will be graded not merely on your final answer, but also on the quality and correctness of the work leading up to it.**
- If you need more space to work a problem, you may use the back of the cover page or the back of the exam. Please indicate where the problem is located.
- **Calculators are not allowed.**
- SCHOLASTIC DISHONESTY WILL NOT BE TOLERATED.

Good Luck!

Check the back of the page for problem.

1. (21 point) Determine whether each is True or False in $\mathbb{R}^3$, i.e. 3-space. Circle your answer.

T **F** The distance from the center of the sphere, $(x-2)^2+(y-4)^2+(z-3)^2 = 25$, to xz−coordinate plane is 3.

$(2, 4, 3)$ ↳ $|y$-value$|$ is distance.

T F The planes $x - y + z = 10$ and $2x + 5y + 3z = 2$ are perpendicular.

$n_1 = \langle 1, -1, 1 \rangle$ $n_2 = \langle 2, 5, 3 \rangle$ $n_1 \cdot n_2 = 2 - 5 + 3 = 0$

T F If $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 4$ then the minimum value of $|\mathbf{a} \times \mathbf{b}|$ is 0.

T **F** If $\mathbf{r_1}(t)$ and $\mathbf{r_2}(s)$ are any two non-parallel lines, then there exists a plane that contains both lines.

T F The line through the points $\overset{A}{(3, 1, 7)}$ and $\overset{B}{(1, 0, 3)}$ is parallel to the line $L_2 : x = 2 + 6t,\ y = 1 + 3t,\ z = 5 + 12t$

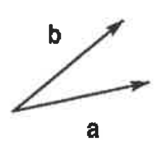
$\vec{AB} = \langle -2, -1, -4 \rangle$

$V = \langle 6, 3, 12 \rangle$ $(-3)\vec{AB} = \vec{V}$

T F The graph of the space curve given by $\mathbf{r}(t) = \langle 2\sin t, 3t, 2\cos t \rangle$ lies on the surface of the circular cylinder $x^2 + z^2 = 4$. ✓

$(2\sin t)^2 + (2\cos t)^2 = 4\sin^2 t + 4\cos^2 t = 4$ ✓

T **F** The vectors $\mathbf{a}$ and $\mathbf{b}$ are drawn on the surface of this paper as shown in the picture. Then $\mathbf{b} \times \mathbf{a}$ will be a vector that is directed above this paper. NOTE: directed above this paper is the same as saying the cross product points towards the ceiling when this paper is on a flat surface.



Check the back of the page for problem.

5. Use the vectors $\mathbf{a} = \langle 1, 1, 0 \rangle$ and $\mathbf{b} = \langle 0, 2, 5 \rangle$ for this problem.

(a) (5 points) Compute $\mathbf{b} \times \mathbf{a}$.

$$b \times a = \begin{vmatrix} i & j & k \\ 0 & 2 & 5 \\ 1 & 1 & 0 \end{vmatrix} = (0-5)i - (0-5)j + (0-2)k$$

$$= \langle -5, 5, -2 \rangle$$

(b) (2 points) Find the area of the parallogram that is created by the vectors $\mathbf{a}$ and $\mathbf{b}$.

$$|b \times a| = \sqrt{25+25+4} = \sqrt{54}$$

6. (6 points) Find cosine of the angle between vector $\mathbf{a} = \langle 5,\ 3,\ 1 \rangle$ and vector $\mathbf{b} = \langle 2,\ -2,\ 1 \rangle$.

$|a| = \sqrt{25+9+1} = \sqrt{35}$

$|b| = \sqrt{4+4+1} = \sqrt{9}$

$$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{5}{3\sqrt{35}}$$

$a \cdot b = 10 - 6 + 1 = 5$

7. (7 points) Find the equation of the plane through the point $(1, 2, 7)$ that will be perpendicular to the line $x = 4 + 5t$, $y = 3 + 7t$, $z = 6 + 4t$.

so $n = v = \langle 5, 7, 4 \rangle$

$$5(x-1) + 7(y-2) + 4(z-7) = 0$$

or

$$5x + 7y + 4z = 47$$

Check the back of the page for problem.

2. (8 points) Find the center and the radius of $x^2 - 4x + y^2 + 8y + z^2 = 50$

$x^2 - 4x + (2)^2 + y^2 + 8y + (4)^2 + z^2 = 50 + 4 + 16$

$(x-2)^2 + (y+4)^2 + z^2 = 70$

Center $(2, -4, 0)$ $R = \sqrt{70}$

3. (6 points) Find the unit tangent vector at $t = 2$ for $\mathbf{r}(t) = \langle 3t,\ t^3 - 2,\ 7 + 2t^2 \rangle$.

$r' = \langle 3, 3t^2, 4t \rangle$

$r'(2) = \langle 3, 12, 8 \rangle$

$|r'(2)| = \sqrt{9 + 144 + 64}$

$= \sqrt{217}$

$T(2) = \frac{1}{\sqrt{217}} \langle 3, 12, 8 \rangle$

4. (6 points) **Set Up The Integral** that will give the length of the arc for the space curve. Do not do the integration.

$\mathbf{r}(t) = \langle t + t^2, 2 + t^3, t^2 - 2 \rangle$ from the point $(2, -6, 2)$ to $(0, 2, -2)$.

$(2, -6, 2)$ $t^2 - 2 = 2$

$t^2 = 4$

$t = \pm 2$

$t = 2$ ✗ $t = -2$ ✓

$t^2 - 2 = -2$

$t^2 = 0$

$t = 0$

$t = -2$

$r' = \langle 1 + 2t, 3t^2, 2t \rangle$

$$\int_{-2}^{0} \sqrt{(1+2t)^2 + 9t^4 + 4t^2}\, dt$$

Check the back of the page for problem.

8. (6 points) Find the equation of the tangent line for the space curve $r(t) = \langle t+3,\ t^2,\ t^3 \rangle$ at the point $(5, 4, 8)$.

$5 = t+3$
$t = 2$

$r' = \langle 1, 2t, 3t^2 \rangle$

$v = r'(2) = \langle 1, 4, 12 \rangle$

Line

$x = 5 + t$
$y = 4 + 4t$
$z = 8 + 12t$

9. Use the space curve to answer the following. $\mathbf{r}(t) = \langle\ \cos(t),\ \sin(t),\ 2t\rangle$

(a) (5 points) Reparameterize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

$r' = \langle -\sin t, \cos t, 2 \rangle$

$|r'| = \sqrt{\sin^2 t + \cos^2 t + 4}$
$= \sqrt{1+4} = \sqrt{5}$

$s = \int_0^t \sqrt{5}\, du = u\sqrt{5} \Big|_0^t$

$s = t\sqrt{5}$

$t = \frac{s}{\sqrt{5}}$

$r(s) = \langle \cos\left(\frac{s}{\sqrt{5}}\right), \sin\left(\frac{s}{\sqrt{5}}\right), \frac{2s}{\sqrt{5}} \rangle$

(b) (2 points) Find the point on the space curve that is a distance of 2 units from the point at $t = 0$ along the path of the space curve in the direction of increasing t.

$s = 2$

$r(2) = \langle \cos\left(\frac{2}{\sqrt{5}}\right), \sin\left(\frac{2}{\sqrt{5}}\right), \frac{4}{\sqrt{5}} \rangle$

point $\left(\cos\left(\frac{2}{\sqrt{5}}\right), \sin\left(\frac{2}{\sqrt{5}}\right), \frac{4}{\sqrt{5}} \right)$

Check the back of the page for problem.

10. (7 points) Find an equation of the line of intersection of these planes.

$2x - y = 2$ $\quad n_1 = \langle 2, -1, 0 \rangle$

$3x - y + 2z = 7$ $\quad n_2 = \langle 3, -1, 2 \rangle$

$$v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 2 & -1 & 0 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= (-2-0)i - (4-0)j + (-2 - -3)k$$

$$= \langle -2, -4, 1 \rangle$$

let $y=0$ $\quad 2x = 2$

$x = 1$

$3(1) - 0 + 2z = 7$

$3 + 2z = 7$

$2z = 4$

$z = 2$

point $(1, 0, 2)$

$$\boxed{\begin{aligned} x &= 1 - 2t \\ y &= 0 - 4t \\ z &= 2 + t \end{aligned}}$$

11. (6 points) Find the distance between the line and the plane.

$r(t) = \langle 2 + 4t,\ 1 + 2t, 1 - 3t \rangle$ $\quad v = \langle 4, 2, -3 \rangle$ $\quad v \cdot n = 4 + 2 - 6 = 0$

$x + y + 2z = 10$ $\quad n = \langle 1, 1, 2 \rangle$ $\quad$ line is parallel to the plane.

$x + y + 2z - 10 = 0$

point $(2, 1, 1)$

$$\text{distance} = \frac{|2 + 1 + 2(1) - 10|}{\sqrt{1 + 1 + 4}} = \frac{|-5|}{\sqrt{6}} = \frac{5}{\sqrt{6}}$$

Check the back of the page for problem.

12. (7 points) Find an equation of a plane that contain these lines. Assume that these lines are not skew.

$\mathbf{r_1}(t) = \langle 1+2t,\ 3+t,\ 5+2t \rangle$ $v_1 = \langle 2, 1, 2 \rangle$ point $(1,3,5)$

$\mathbf{r_2}(t) = \langle 1+t,\ 2t,\ 1+3t \rangle$ $v_2 = \langle 1, 2, 3 \rangle$ $(1,0,1)$

$$v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = (3-4)i - (6-2)j + (4-1)k = \langle -1, -4, 3 \rangle$$

$$\boxed{-(x-1) - 4(y-3) + 3(z-5) = 0}$$

$$-x - 4y + 3z = -1 - 12 + 15 = 2$$

$$\boxed{-x - 4y + 3z = 2}$$

or

$$\boxed{x + 4y - 3z = -2}$$

13. (6 points) Let $\mathbf{r}(t) = \langle 2t+5,\ t^2+4,\ t^3 \rangle$. Compute the curvature at $t = 1$.

$r' = \langle 2, 2t, 3t^2 \rangle$ $|r'| = \sqrt{4 + 4t^2 + 9t^4}$

$r'' = \langle 0, 2, 6t \rangle$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 2 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = (12t^2 - 6t^2)i - (12t - 0)j + (4-0)k$$

$$= \langle 6t^2, -12t, 4 \rangle$$

$$K = \frac{\sqrt{36t^4 + 144t^2 + 16}}{\left(\sqrt{4 + 4t^2 + 9t^4}\right)^3} \qquad K(1) = \frac{\sqrt{36 + 144 + 16}}{\left(\sqrt{4+4+9}\right)^3} = \frac{\sqrt{196}}{(\sqrt{17})^3} = \frac{14}{(17)^{3/2}}$$

Check the back of the page for problem.