Math 251 | Exam 1 B | Thursday, February 8, 2024

Printed Name: ______________________

Section: Key

UIN: ______________________

Signature: ______________________

On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work.

Section 507: class time TR 12:45
Section 508: class time TR 2:20

- **Show all appropriate work to receive full credit.**
- **You will be graded not merely on your final answer, but also on the quality and correctness of the work leading up to it.**
- If you need more space to work a problem, you may use the back of the cover page or the back of the exam. Please indicate where the problem is located.
- **Calculators are not allowed.**
- SCHOLASTIC DISHONESTY WILL NOT BE TOLERATED.

Good Luck!

Check the back of the page for problem.

1. (21 point) Determine whether each is True or False in $\mathbb{R}^3$, i.e. 3-space. Circle your answer.

~~T F The distance from the center of the sphere, $(x-2)^2+(y-4)^2+(z-3)^2 = 25$, to xz−coordinate plane is 3.~~

T F The planes $x - y + z = 10$ and $2x + 5y + 3z = 2$ are perpendicular.

T F If $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 4$ then the minimum value of $|\mathbf{a} \times \mathbf{b}|$ is 0.

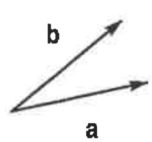
T F If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ are any two non-parallel lines, then there exists a plane that contains both lines.

T F The line through the points $(3, 1, 7)$ and $(1, 0, 3)$ is parallel to the line $L_2 : x = 2 + 6t,\ y = 1 + 3t,\ z = 5 + 12t$

T F The graph of the space curve given by $\mathbf{r}(t) = \langle 2\sin t, 3t, 2\cos t\rangle$ lies on the surface of the circular cylinder $x^2 + z^2 = 4$.

T F The distance from the center of the sphere, $(x-2)^2+(y-4)^2+(z-3)^2 = 25$, to xz−coordinate plane is 3.

T F The vectors $\mathbf{a}$ and $\mathbf{b}$ are drawn on the surface of this paper as shown in the picture. Then $\mathbf{b} \times \mathbf{a}$ will be a vector that is directed above this paper. NOTE: directed above this paper is the same as saying the cross product points towards the ceiling when this paper is on a flat surface.



Check the back of the page for problem.

2. (8 points) Find the center and the radius of $x^2 + 9x + y^2 + z^2 - 20z = 60$

$$x^2 + 9x + \left(\frac{9}{2}\right)^2 + y^2 + z^2 - 20z + (10)^2 = 60 + \left(\frac{9}{2}\right)^2 + 100$$

$$\left(x + \frac{9}{2}\right)^2 + y^2 + (z-10)^2 = 60 + \frac{81}{4} + 100 = 160 + \frac{81}{4}$$

Center $\left(-\frac{9}{2}, 0, 10\right)$ $R = \sqrt{160 + \frac{81}{4}}$

3. (6 points) Find the unit tangent vector at $t = 2$ for $\mathbf{r}(t) = \langle t^3 + 1,\ 1 + t,\ 7 - t^2 \rangle$.

$r' = \langle 3t^2, 1, -2t \rangle$

$r'(2) = \langle 12, 1, -4 \rangle$

$|r'(2)| = \sqrt{144 + 1 + 16} = \sqrt{161}$

$$T(2) = \frac{1}{\sqrt{161}} \langle 12, 1, -4 \rangle$$

4. (6 points) **Set Up The Integral** that will give the length of the arc for the space curve. Do not do the integration.

$\mathbf{r}(t) = \langle t^3 + 1,\ 5 + t^4,\ 2t + 10 \rangle$ from the point $(2, 6, 12)$ to $(9, 21, 14)$.

$2t + 10 = 12$

$2t = 2$

$t = 1$

$2t + 10 = 14$

$2t = 4$

$t = 2$

$r' = \langle 3t^2, 4t^3, 2 \rangle$

$$\int_1^2 \sqrt{9t^4 + 16t^6 + 4}\, dt$$

5. Use the vectors $\mathbf{a} = \langle 1, 1, 0\rangle$ and $\mathbf{b} = \langle 0, 3, 10\rangle$ for this problem.

(a) (5 points) Compute $\mathbf{b} \times \mathbf{a}$.

$$|b \times a| = \begin{vmatrix} i & j & k \\ 0 & 3 & 10 \\ 1 & 1 & 0 \end{vmatrix} = (0-10)i - (0-10)j + (0-3)k$$

$$= \langle -10, 10, -3 \rangle$$

(b) (2 points) Find the area of the parallogram that is created by the vectors $\mathbf{a}$ and $\mathbf{b}$.

$$|b \times a| = \sqrt{100+100+9} = \sqrt{209}$$

6. (6 points) Find cosine of the angle between vector $\mathbf{a} = \langle 6, 4, 2\rangle$ and vector $\mathbf{b} = \langle 2, 1, -1\rangle$.

$|a| = \sqrt{36+16+4} = \sqrt{56}$

$|b| = \sqrt{4+1+1} = \sqrt{6}$

$a \cdot b = 12 + 4 + -2 = 14$

$$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{14}{\sqrt{6}\sqrt{56}}$$

7. (7 points) Find the equation of the plane through the point $(2, 5, 9)$ that will be perpendicular to the line $x = 1 + 3t$, $y = 2 + t$, $z = 7 + 2t$.

$n = v = \langle 3, 1, 2 \rangle$

$$3(x-2) + 1(y-5) + 2(z-9) = 0$$

$$3x + y + 7z = 29$$

Check the back of the page for problem.

8. (6 points) Find the equation of the tangent line for the space curve $r(t) = \langle t^3,\ t^2 - 1,\ t + 1\rangle$ at the point $(27, 8, 4)$.

$r' = \langle 3t^2, 2t, 1\rangle$

$t+1=4$

$t=3$

$V = r'(3) = \langle 27, 6, 1\rangle$

Line

$x = 27 + 27t$

$y = 8 + 6t$

$z = 4 + t$

9. Use the space curve to answer the following. $\mathbf{r}(t) = \langle\ \cos(t),\ \sin(t),\ 5t\rangle$

(a) (5 points) Reparameterize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

$r' = \langle -\sin t, \cos t, 5\rangle$

$|r'| = \sqrt{\sin^2 t + \cos^2 t + 25}$

$= \sqrt{1+25} = \sqrt{26}$

$s = \int_0^t \sqrt{26}\,du = \sqrt{26}\,u\Big|_0^t$

$s = t\sqrt{26}$

$t = \frac{s}{\sqrt{26}}$

$r(s) = \langle \cos\left(\frac{s}{\sqrt{26}}\right), \sin\left(\frac{s}{\sqrt{26}}\right), \frac{5s}{\sqrt{26}}\rangle$

(b) (2 points) Find the point on the space curve that is a distance of 2 units from the point at $t = 0$ along the path of the space curve in the direction of increasing t.

$s = 2$

$r(2) = \langle \cos\left(\frac{2}{\sqrt{26}}\right), \sin\left(\frac{2}{\sqrt{26}}\right), \frac{10}{\sqrt{26}}\rangle$

point $\left(\cos\left(\frac{2}{\sqrt{26}}\right), \sin\left(\frac{2}{\sqrt{26}}\right), \frac{10}{\sqrt{26}}\right)$

Check the back of the page for problem.

10. (7 points) Find an equation of the line of intersection of these planes.

$4x - y = 12$ $\quad n_1 = \langle 4, -1, 0 \rangle$

$2x - y + 2z = 2$ $\quad n_2 = \langle 2, -1, 2 \rangle$

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 4 & -1 & 0 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= (-2-0)i - (8-0)j + (-4 - -2)k$$

$$= \langle -2, -8, -2 \rangle$$

$y=0$ $\quad 4x - 0 = 12$

$x = 3$

$2(3) - 0 + 2z = 2$

$2z = -4$

$z = -2$

point $(3, 0, -2)$

$$\boxed{\begin{aligned} x &= 3 - 2t \\ y &= 0 - 8t \\ z &= -2 - 2t \end{aligned}}$$

11. (6 points) Find the distance between the line and the plane.

$r(t) = \langle 5 + 3t,\ 2 + t,\ -3t \rangle$ $\quad v = \langle 3, 1, -3 \rangle$

$x + 3y + 4z = 20$ $\quad n = \langle 1, 3, 4 \rangle$

$v \cdot n = 3 + 3 - 12 = -6$

v not perp to n

so line is not parallel to the plane re intersects

distance = 0

Check the back of the page for problem.

12. (7 points) Find an equation of a plane that contain these lines. Assume that these lines are not skew.

$\mathbf{r_1}(t) = \langle 3+2t,\ 2+t,\ 4+4t\rangle$ $V_1 = \langle 2,1,4\rangle$ pt $(3,2,4)$

$\mathbf{r_2}(t) = \langle 2+t,\ 3t-6,\ 5+t\rangle$ $V_2 = \langle 1,3,1\rangle$ $(2,-6,5)$

$$V_1 \times V_2 = \begin{vmatrix} i & j & k \\ 2 & 1 & 4 \\ 1 & 3 & 1 \end{vmatrix} = (1-12)i - (2-4)j + (6-1)k = \langle -11, 2, 5\rangle$$

$$\boxed{-11(x-3) + 2(y-2) + 5(z-4) = 0}$$

or

$$\boxed{\begin{array}{c} -11x + 2y + 5z = -9 \\ \hline 11x - 2y - 5z = 9 \end{array}}$$

13. (6 points) Let $\mathbf{r}(t) = \left\langle 3t+2,\ \frac{1}{2}t^2+7,\ \frac{1}{4}t^4\right\rangle$. Compute the curvature at $t=1$.

$r' = \langle 3, t, t^3\rangle$ $|r'| = \sqrt{9+t^2+t^6}$

$r'' = \langle 0, 1, 3t^2\rangle$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 3 & t & t^3 \\ 0 & 1 & 3t^2 \end{vmatrix} = (3t^3 - t^3)i - (9t^2 - 0)j + (3-0)k$$

$$= \langle 2t^3, -9t^2, 3\rangle$$

$$K = \frac{\sqrt{4t^6 + 81t^4 + 9}}{(9+t^2+t^6)^{3/2}}$$

$$K(1) = \frac{\sqrt{4+81+9}}{(9+1+1)^{3/2}} = \frac{\sqrt{94}}{(11)^{3/2}}$$

Check the back of the page for problem.