Problem 1

Let $\mathbf{r}(t) = \langle 5-t, 4-3t, 3t \rangle$ and $P(4, 1, 3)$

(a) Find the arc length function for the curve measured from the point $P$ in the direction of increasing $t$.

(b) Reparameterize the curve with respect to the arc length starting from $P$.

(c) Find the point 4 units along the curve (in the direction of increasing $t$) from $P$.

point $P$ is when $t = 1$

$r' = \langle -1, -3, 3 \rangle$

$$s = \int_1^t |r'(m)|\, dm = \int_1^t \sqrt{(-1)^2 + (-3)^2 + 3^2}\, dm$$

$$= \int_1^t \sqrt{1+9+9}\, dm = \int_1^t \sqrt{19}\, dm = \sqrt{19}\, m \Big|_1^t$$

$$s = t\sqrt{19} - \sqrt{19}$$

b) $s + \sqrt{19} = t\sqrt{19}$

$$t = \frac{s+\sqrt{19}}{\sqrt{19}}$$

$$r(s) = \left\langle 5 - \frac{s+\sqrt{19}}{\sqrt{19}},\ 4 - 3\left(\frac{s+\sqrt{19}}{\sqrt{19}}\right),\ 3\left(\frac{s+\sqrt{19}}{\sqrt{19}}\right) \right\rangle$$

c) $s = 4$ find $r(4)$

$$r(4) = \left\langle 5 - \frac{4+\sqrt{19}}{\sqrt{19}},\ 4 - 3\left(\frac{4+\sqrt{19}}{\sqrt{19}}\right),\ 3\left(\frac{4+\sqrt{19}}{\sqrt{19}}\right) \right\rangle$$

$r(4) \approx \langle 3.0823, -1.7530, 5.7530 \rangle$

Problem 2

Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6, 18, 36)$

Let $x = t$ Then $y = \frac{t^2}{2}$ and $z = \frac{1}{3}\left(t \cdot \frac{t^2}{2}\right)$

$= \frac{1}{6} t^3$

$r(t) = \langle t, \frac{t^2}{2}, \frac{t^3}{6} \rangle$

The origin is when $t = 0$ and $(6, 18, 36)$ is when $t = 6$

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$r'(t) = \langle 1, t, \frac{1}{2} t^2 \rangle$

arc length $= \int_0^6 \sqrt{1^2 + t^2 + \left(\frac{1}{2} t^2\right)^2}\, dt$

$= \int_0^6 \sqrt{1 + t^2 + \frac{t^4}{4}}\, dt$

$= \int_0^6 \sqrt{\frac{4 + 4t^2 + t^4}{4}}\, dt$

$= \int_0^6 \frac{\sqrt{t^4 + 4t^2 + 4}}{2}\, dt = \int_0^6 \frac{\sqrt{(t^2+2)^2}}{2}\, dt$

$$= \int_0^6 \frac{1}{2}\left(t^2+2\right)\, dt = \int_0^6 \frac{1}{2}t^2 + 1 \, dt$$

$$= \left(\frac{1}{6}t^3 + t\right)\Big|_0^6$$

$$= \frac{1}{6}\cdot 6^3 + 6 - (0) = 36 + 6 = \underline{\underline{42}}$$

Problem 3

Find the arc length function for $\mathbf{r}(t) = \langle e^t, e^t \sin(t), e^t \cos(t) \rangle$ from the point $(1, 0, 1)$ in the direction of increasing $t$.

$\hookrightarrow t = 0$

$$r' = \langle e^t, e^t \sin(t) + e^t \cos(t), e^t \cos(t) - e^t \sin(t) \rangle$$

$$|r'| = \sqrt{(e^t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2}$$

$$= \sqrt{(e^t)^2 + \left[e^t(\sin t + \cos t)\right]^2 + \left[e^t(\cos t - \sin t)\right]^2}$$

$$= \sqrt{(e^t)^2 + (e^t)^2(\sin t + \cos t)^2 + (e^t)^2(\cos(t) - \sin(t))^2}$$

$$= \sqrt{(e^t)^2 \cdot \left[1 + \sin^2 t + 2\sin t \cos t + \cos^2 t + \cos^2 t - 2\sin(t)\cos(t) + \sin^2 t\right]}$$

$$= e^t \sqrt{1 + 2\sin^2 t + 2\cos^2 t}$$

$$= e^t \sqrt{1 + 2\left[\sin^2 t + \cos^2 t\right]} = e^t \sqrt{1 + 2(1)} = e^t \sqrt{3}$$

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$$s = \int_0^t |r'(m)|\, dm = \int_0^t e^m \sqrt{3}\, dm = \sqrt{3} e^m \Big|_0^t$$

$$\boxed{s = \sqrt{3} e^t - \sqrt{3}}$$

Problem 4

Find the curvature of $\mathbf{r}(t) = \langle t^3, t^2, t \rangle$ at a general point and then at $(8, 4, 2)$.

$t = 2$ at the point.

$r' = \langle 3t^2, 2t, 1 \rangle$ $\quad |r'| = \sqrt{9t^4 + 4t^2 + 1}$

Lets try the formula $\frac{|T'|}{|r'|}$

$$T = \frac{1}{\sqrt{9t^4+4t^2+1}} \langle 3t^2, 2t, 1 \rangle$$

use product Rule.

$$T' = -\tfrac{1}{2}(9t^4+4t^2+1)^{-3/2} \cdot (36t^3+8t) \langle 3t^2, 2t, 1 \rangle + \frac{1}{\sqrt{9t^4+4t^2+1}} \langle 6t, 2, 0 \rangle$$

now we need to find the magnitude of $T'$

... nope not going to do this

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Lets do $\frac{|r' \times r''|}{|r'|^3}$

$|r'| = \sqrt{9t^4 + 4t^2 + 1}$

$r' = \langle 3t^2, 2t, 1 \rangle$

$r'' = \langle 6t, 2, 0 \rangle$

note if you only want curvature at $t = 2$ (the point $(8,4,2)$) then evaluate $r'$ and $r''$ at $t = 2$ and then do the cross product.

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 3t^2 & 2t & 1 \\ 6t & 2 & 0 \end{vmatrix}$$

$$= \langle -2, -(0-6t), 6t^2 - 12t^2 \rangle$$

$$= \langle -2, 6t, -6t^2 \rangle$$

$r'(2) = \langle 12, 4, 1 \rangle$

$r''(2) = \langle 12, 2, 0 \rangle$

$$r'(2) \times r''(2) = \begin{vmatrix} i & j & k \\ 12 & 4 & 1 \\ 12 & 2 & 0 \end{vmatrix}$$

$$= \langle -2, -(0-12), 24-48 \rangle$$

$$= \langle -2, 12, -24 \rangle$$

$$\kappa = \frac{|r' \times r''|}{|r'|^3} = \frac{\sqrt{4 + 36t^2 + 36t^4}}{(\ldots)^{3/2}}$$

$r'(2) = \sqrt{144 + 16 + 1}$

$$K = \frac{|r' \times r''|}{|r'|^3} = \frac{\sqrt{4 + 36t^2 + 36t^4}}{\left(9t^4 + 4t^2 + 1\right)^{3/2}}$$

at $t=2$

$$K = \frac{\sqrt{4 + 144 + 576}}{\left(144 + 16 + 1\right)^{3/2}}$$

$$K = \frac{\sqrt{724}}{(161)^{3/2}}$$

$$r'(2) = \sqrt{144 + 16 + 1}$$

$$K\Big|_{t=2} = \frac{\sqrt{4 + 144 + 576}}{(161)^{3/2}}$$

$$K\Big|_{t=2} = \frac{\sqrt{724}}{(161)^{3/2}}$$