Problem 1

If $f(x,y) = x^4 + 4x^3y - y^3$, find the gradient of f and evaluate the gradient at the point $P(1,2)$

$$\nabla f = \langle f_x , f_y \rangle$$

$$= \langle 4x^3 + 12x^2y , 4x^3 - 3y^2 \rangle$$

$$\nabla f(1,2) = \langle 4(1)^3 + 12(1)^2(2) , 4(1)^3 - 3(2)^2 \rangle$$

$$= \langle 4 + 24 , 4 - 12 \rangle$$

$$= \langle 28 , -8 \rangle$$

Problem 2

Given $g(x, y, z) = x^2 + 4yz^3 + 2x^4z + z^2$.

(a) Find the gradient vector at (3,2,1).

(b) Find the directional derivative at (3,2,1), in the direction of $\langle 4, 4, 2\rangle$.

(c) Find the maximum value of the directional derivative at (3,2,1).

(d) Find the direction of greatest decrease at (3,2,1).

A) $\nabla g = \langle g_x, g_y, g_z \rangle$

$= \langle 2x + 8x^3 z,\ 4z^3,\ 12yz^2 + 2x^4 + 2z \rangle$

$\nabla g(3,2,1) = \langle 2(3) + 8(3)^3(1),\ 4(1)^3,\ 12(2)(1)^2 + 2(3)^4 + 2(1) \rangle$

$= \langle 6 + 216,\ 4,\ 24 + 162 + 2 \rangle$

$= \langle 222, 4, 188 \rangle$

b) $|\langle 4,4,2\rangle| = \sqrt{16+16+4} = \sqrt{36} = 6$

unit vector. $u = \frac{1}{6}\langle 4, 4, 2\rangle = \langle \frac{4}{6}, \frac{4}{6}, \frac{2}{6} \rangle = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$

$D_u g(3,2,1) = \nabla g \cdot u$

$= \langle 222, 4, 188 \rangle \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$

$= \frac{444}{3} + \frac{8}{3} + \frac{188}{3} = \frac{640}{3}$

c) max value $= |\nabla g(3,2,1)|$

$= \sqrt{222^2 + 4^2 + 188^2}$

≈ 290.936

d) direction of greatest decrease $= -\nabla g(3,2,1)$

$= \langle -222, -4, -188 \rangle$

Problem 3

Consider the surface given by $x^2 + 2y^2 + z^3 = 10$. Find an equation for the tangent plane and the equation for the normal line to the surface at the point $(1, 1, 2)$.

$$F(x,y,z) = x^2 + 2y^2 + z^3 - 10$$

$F_x = 2x$

$F_y = 4y$

$F_z = 3z^2$

$$n = \nabla F = \langle 2x, 4y, 3z^2 \rangle$$

at the point $(1,1,2)$

$$n = \langle 2, 4, 12 \rangle$$

tangent plane:

$$\boxed{2(x-1) + 4(y-1) + 12(z-2) = 0}$$

normal line:

$$\boxed{\begin{aligned} x &= 1+2t \\ y &= 1+4t \\ z &= 2+12t \end{aligned}}$$

Problem 4

Find all points at which the direction of fastest change of the function $f(x, y)$ is $\mathbf{i}+\mathbf{j}$.

$f(x, y) = x^2 + y^2 - 2x - 6y$

This says that $\nabla f = m\langle 1, 1\rangle$ with $m > 0$

$\nabla f = \langle 2x-2, 2y-6\rangle$

So $2x - 2 = m$ and $2y - 6 = m$

note for $m > 0$ need $2x-2>0$, $2x>2$, $x>1$ and $2y-6>0$, $2y>6$, $y>3$

Thus $2x - 2 = 2y - 6$

$2x = 2y - 4$

$x = y - 2$

so all points on the line $x = y - 2$ or $y = x + 2$ such that $y > 3$ and $x > 1$ will work.