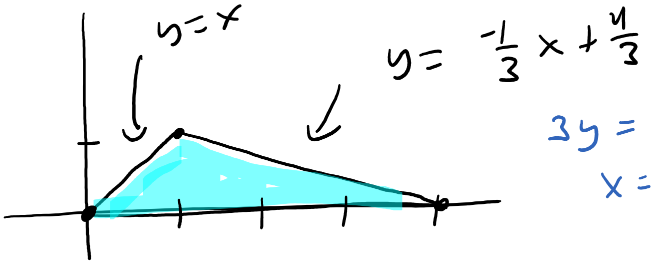
Problem 1

Find the center of mass of the lamina that occupies the triangle region with vertices $(0,0)$, $(1,1)$, and $(4,0)$. The density of the region is given by $\rho(x,y) = y$



$3y = -x + 4$

$x = 4 - 3y$

Based on the Region I want all Integrals to be $dx\,dy$

$0 \le y \le 1$

$y \le x \le 4 - 3y$

$$\text{mass} = M = \iint_D \rho(x,y)\, dA = \int_{y=0}^{1} \int_{x=y}^{4-3y} y \, dx\, dy$$

$$= \int_{y=0}^{1} xy \Big|_{x=y}^{4-3y} dy$$

$$= \int_{y=0}^{1} y(4-3y) - y(y)\, dy = \int_{y=0}^{1} 4y - 4y^2\, dy$$

$$= \left(2y^2 - \frac{4}{3}y^3\right)\Big|_0^1 = 2 - \frac{4}{3} = \frac{2}{3}$$

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now we need the moments M_x + M_y

$$M_x = \iint y\,\rho(x,y)\, dA = \int^{1} \int^{4-3y} y \cdot y\, dx\, dy$$

$$M_x = \iint_D y\,\rho(x,y)\,dA = \int_{y=0} \int_{x=y} y\cdot y\,dx\,dy$$

$$= \int_{y=0}^{1} y^2 x \Big|_{x=y}^{4-3y} dy = \int_{y=0}^{1} y^2(4-3y) - y^2(y)\,dy$$

$$= \int_{y=0}^{1} 4y^2 - 4y^3\,dy = \left(\frac{4y^3}{3} - y^4\right)\Big|_0^1$$

$$= \frac{4}{3} - 1 - (0) = \frac{1}{3}$$

$$M_y = \iint_D x\,\rho(x,y)\,dA = \int_{y=0}^{1} \int_{x=y}^{4-3y} xy\,dx\,dy$$

$$= \int_{y=0}^{1} \frac{1}{2}x^2 y \Big|_{x=y}^{4-3y} dy = \int_{y=0}^{1} \frac{1}{2}y(4-3y)^2 - \frac{1}{2}y(y)^2\,dy$$

$$= \int_{y=0}^{1} \frac{1}{2}y(16 - 24y + 9y^2) - \frac{1}{2}y^3\,dy$$

$$= \int^{1} \ldots + \frac{9}{\ldots}y^3 - \frac{1}{\ldots}y^3\,dy$$

$$= \int^{1} 8y - 12y^2 + \frac{9}{2}y^3 - \frac{1}{2}y^3 \, dy$$

$$= \int 8y - 12y^2 + 4y^3 \, dy$$

$$= 4y^2 - 4y^3 + y^3 \Big) \Big|_{y=0}^{1} = 4 - 4 + 1 = 1$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

center of mass $= (\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{1}{2} \right)$

Problem 2

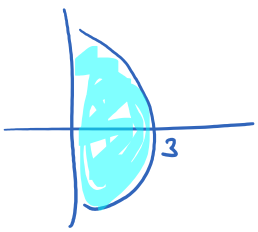
Find the center of mass for the region $D = \{(x, y) \mid x^2 + y^2 \leq 9, \quad x \geq 0\}$.
The density at any point is proportional to the square of its distance from the origin.

$\rho(x,y) = K \cdot (\text{distance})^2$

This is for the proportional part of the statement

$$\rho(x,y) = K\left(\sqrt{(x-0)^2+(y-0)^2}\right)^2$$

$$\rho(x,y) = K(x^2+y^2)$$



polar Region

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$0 \leq r \leq 3$

$$\text{mass} = m = \iint_D K(x^2+y^2)\,dA = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \int_{r=0}^{3} K r^2 \cdot r\,dr\,d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} 1\,d\theta \cdot \int_{r=0}^{3} K r^3\,dr$$

by Fubini

$$= \theta\Big|_{-\frac{\pi}{2}}^{\pi/2} \cdot \frac{K}{4} r^4\Big|_0^3$$

$$= \left(\frac{\pi}{2} - -\frac{\pi}{2}\right) \cdot \frac{K}{4}(3)^4 = \pi \cdot \frac{K}{4} \cdot 81$$

$$m = \frac{81K\pi}{4}$$

$$M_x = \iint_D y\,\rho(x,y)\,dA = \iint_D yK(x^2+y^2)\,dA$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \int_{r=0}^{3} r\sin\theta\; K\; r^2 \cdot r\,dr\,d\theta$$

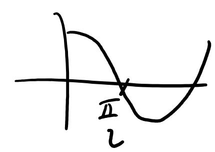
$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\,d\theta \cdot \int_{r=0}^{3} K r^4\,dr$$

by Fubini

$$= -\cos\theta \Big|_{-\frac{\pi}{2}}^{\pi/2} \cdot \frac{K}{5} r^5 \Big|_0^3$$

$$= \left(-\cos\frac{\pi}{2} - -\cos\left(-\frac{\pi}{2}\right)\right) \cdot \frac{K}{5} \cdot 3^5$$

$$= (0 + 0)\,\frac{K}{5}\,3^5 = 0$$



$$M_y = \iint_D x \cdot K\,(x^2+y^2)\,dA$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \int_{r=0}^{3} r\cos\theta \cdot K r^2 \cdot r\,dr\,d\theta$$

$$= \int^{\pi/2} \cos\theta\,d\theta \cdot \int^{3} K r^4\,dr$$

$$= \int_{\theta=-\frac{\pi}{2}} \cos\theta \, d\theta \cdot \int_{r=0} K r^4 \, dr$$

$$= \sin\theta \Big|_{\theta=-\frac{\pi}{2}}^{\pi/2} \cdot \frac{K}{5} r^5 \Big|_0^3$$

$$= \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right)\right) \cdot \frac{K}{5}(3)^5$$

$$= (1 - -1) \cdot \frac{K\,3^5}{5} = \frac{2K3^5}{5}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{2K \cdot 3^5}{5}}{\frac{81K\pi}{4}} = \frac{2\cancel{K} \cdot \cancel{3^5}}{5} \cdot \frac{4}{\cancel{81}\cancel{K}\pi}$$

$81 = 3^4$

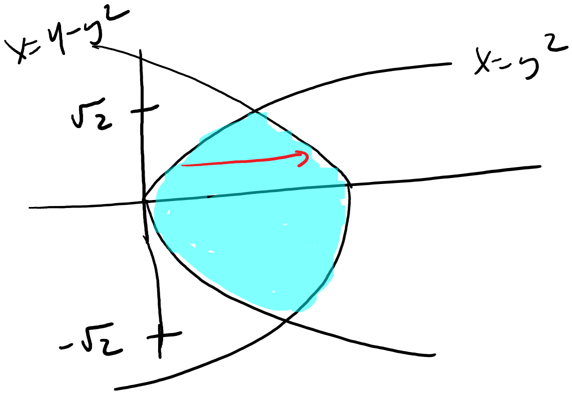
$$= \frac{8 \cdot 3}{5 \cdot \pi} = \frac{24}{5\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{0}{m} = 0$$

$$\text{center of mass} = (\bar{x}, \bar{y}) = \left(\frac{24}{5\pi}, 0\right)$$

Problem 3

Set up the integral to find the mass of a thin lamina bounded by the curves $x = y^2$, $x = 4 - y^2$ with density proportional to the distance from the point $(3, 1)$



$\rho(x,y) = K\sqrt{(x-3)^2 + (y-1)^2}$

lets use a $dx\,dy$ Integral

$-\sqrt{2} \le y \le \sqrt{2}$

$y^2 \le x \le 4 - y^2$

$y^2 = 4 - y^2$

$2y^2 = 4$

$y^2 = 2$

$y = \pm\sqrt{2}$

$$\text{mass} = \int_{y=-\sqrt{2}}^{\sqrt{2}} \int_{x=y^2}^{4-y^2} K\sqrt{(x-3)^2 + (y-1)^2}\,dx\,dy$$

Note symmetry can <u>not</u> be used since the density function is <u>not</u> symetric on Region D.