Problem 1

Let $B = [0,1] \times [1,3] \times [0,2]$. Evaluate

$$\iiint_B x(y+2z)\,dV = \int_{x=0}^{1}\int_{y=1}^{3}\int_{z=0}^{2} x(y+2z)\,dz\,dy\,dx$$

Since all integration limits are numbers and the function can be separated into $g(x)=x$ and $h(y,z)=y+2z$

$$f(x,y,z) = x(y+2z) = g(x)\cdot h(y,z)$$

we can separate the integrals. by Fubini

$$\int_{x=0}^{1} x\,dx \cdot \int_{y=1}^{3}\int_{z=0}^{2} (y+2z)\,dz\,dy$$

$$= \frac{x^2}{2}\Big|_0^1 \cdot \int_{y=1}^{3} (yz+z^2)\Big|_{z=0}^{2}\,dy$$

$$= \frac{1}{2}\cdot\int_{y=1}^{3} 2y+4\,dy = \frac{1}{2}\left[y^2+4y\right]\Big|_1^3$$

$$= \frac{1}{2}\left[9+12-(1+4)\right] = \frac{1}{2}(21-5)$$

$$= \frac{16}{2} = 8$$

Problem 2

$$\text{Evaluate: } \int_0^1 \int_x^{2x} \int_0^{x+y} 6xy \, dz\,dy\,dx = \int_{x=0}^{1} \int_{y=x}^{2x} 6xyz \Big|_{z=0}^{x+y} dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=x}^{2x} 6xy(x+y) \, dy\,dx = \int_{x=0}^{1} \int_{y=x}^{2x} 6x^2y + 6xy^2 \, dy\,dx$$

$$= \int_{x=0}^{1} \left(3x^2y^2 + 2xy^3\right)\Big|_{x}^{2x} dx$$

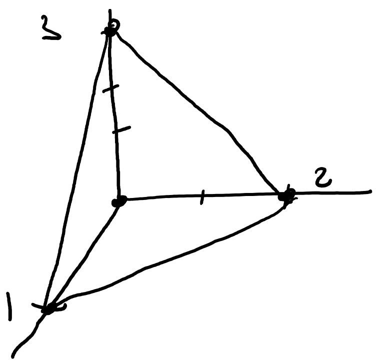
$$= \int_{x=0}^{1} 3x^2(2x)^2 + 2x(2x)^3 - \left[3x^2(x)^2 + 2x(x)^3\right] dx$$

$$= \int_{x=0}^{1} 12x^4 + 16x^4 - 3x^4 - 2x^4 \, dx = \int_{x=0}^{1} 23x^4 \, dx$$

$$= \frac{23x^5}{5}\Big|_0^1 = \frac{23}{5}$$

Problem 3

Setup $\iiint_E xy\, dV$ where V is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.



Top function is the plane formed by the points $(1,0,0)$ $(0,2,0)$ and $(0,0,3)$

We need the equation of this plane.

method 1 use the points to create 2 vectors. Then take the crossproduct to find a normal vector and thus set the equation of the plane.

method 2 we know the points are on a plane that has an equation of $ax+by+cz=d$ with $a,b,c,+d$ are constants. Plugging in the points gives a system of equations.

$(1,0,0) \longrightarrow a(1)+b(0)+c(0)=d \rightarrow a=d$

$(0,2,0) \longrightarrow 2b=d$

$(0,0,3) \longrightarrow 3c=d$

now pick a number for d, I choose $d=6$, and solve for $a, b + c$.

$a = 6$, $b = 3$, $c = 2$

equation is $6x + 3y + 2z = 6$

or $z = \frac{1}{2}\left[6 - 6x - 3y\right]$

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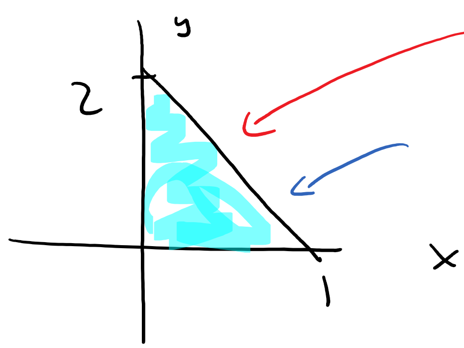
for this problem I'm going to project the solid onto the xy plane:

top function $z = \frac{1}{2}\left[6 - 6x - 3y\right]$

Bottom function $z = 0$

Region D



The equation of this line is when $z=0$ for the plane $6x+3y+2z=6$

$6x + 3y = 6$

or $2x + y = 3$

$y = 3 - 2x$

$0 \le x \le 1$

$0 \le y \le 3 - 2x$

$$\iiint_E xy\, dV = \int_{x=0}^{1} \int_{y=0}^{3-2x} \int_{z=0}^{\frac{1}{2}[6-6x-3y]} xy\; dz\, dy\, dx$$

Problem 4

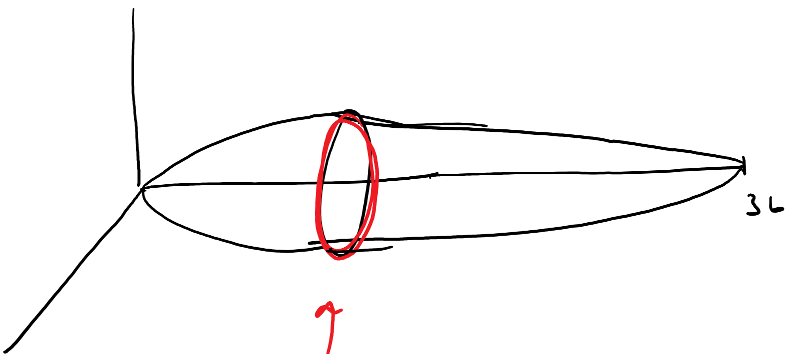
A solid E in enclosed by the paraboloids $y = 3x^2 + 3z^2$ and $y = 36 - x^2 - z^2$. Evaluate

$$\iiint_E x^2 \, dV$$

project on the xz plane.

Left: $y = 3x^2 + 3z^2$

Right: $y = 36 - x^2 - z^2$



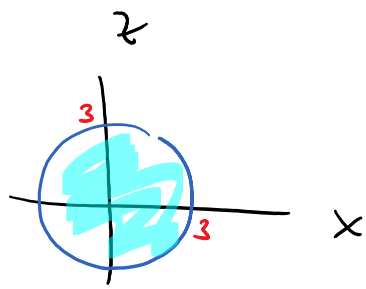
where they intersect

$$36 - x^2 - z^2 = 3x^2 + 3z^2$$

$$36 = 4x^2 + 4z^2$$

$$9 = x^2 + z^2$$

This circle on the xz plane will be the projection ie Region D.



Region D.

$$-3 \le x \le 3$$

$$-\sqrt{9 - x^2} \le z \le \sqrt{9 - x^2}$$

$$\iiint_E x^2 \, dV = \int_{x=-3}^{3} \int_{z=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{y=3x^2+3z^2}^{36-x^2-z^2} x^2 \, dy \, dz \, dx$$

Note this does not look like fun. So lets break this into steps.

$$\iiint x^2 \, dv = \iint \left[\int^{36-x^2-z^2} x^2 \, dy \right] dA$$

$$\iiint_E x^2\,dV = \iint_D \left[\int_{y=3x^2+3z^2} x^2\,dy \right] dA$$

$$= \iint_D x^2 y \Big|_{3x^2+3z^2}^{36-x^2-z^2} dA$$

$$= \iint_D x^2 \left[36 - x^2 - z^2 - (3x^2 + 3z^2) \right] dA$$

$$= \iint_D x^2 \left[36 - 4(x^2 + z^2) \right] dA$$

we have done the y-integral and are left with the double Integral over Region D. This is a polar double Integral.

$0 \le r \le 3$ $\qquad x = r\cos\theta$ $\qquad x^2 + z^2 = r^2$

$0 \le \theta \le 2\pi$ $\qquad z = r\sin\theta$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{3} r^2 \cos^2\theta \left[36 - 4r^2 \right] r\,dr\,d\theta$$

$$= \int^{2\pi} \cdots \int^{3} r^3 (36 - 4r^2)\,dr$$

$$= \int_{\theta=0}^{2\pi} \cos^2\theta \, d\theta \cdot \int_{r=0}^{3} r^3(36-4r^2)\, dr$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{2}\left[1+\cos 2\theta\right] d\theta \cdot \int_{r=0}^{3} 36r^3 - 4r^5 \, dr$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} \cdot \left(9r^4 - \frac{4r^6}{6}\right)\Big|_0^3$$

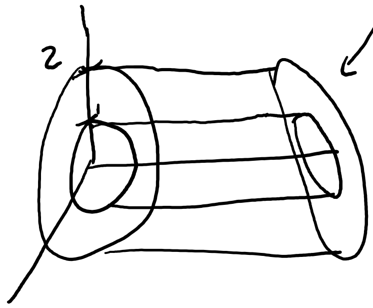
$$= \frac{1}{2}\left[2\pi + \frac{1}{2}\sin(4\pi) - \left(0 + \frac{1}{2}\sin(0)\right)\right] \cdot \left[9\cdot(3)^4 - \frac{2}{3}(3)^6 - 0\right]$$

$$= \frac{1}{2}\left[2\pi + 0 - 0\right] \cdot \left[9 \cdot 3^4 - 2\cdot(3)^5\right]$$

$$= \pi\left[9\cdot(3)^4 - 2\cdot(3)^5\right] = \dots = \pi\cdot(3)^5$$

Problem 5

Set up the integral that would compute the volume of the solid between the cylinders $x^2 + z^2 = 4$ and $x^2 + z^2 = 1$ and bounded by the planes $y = x + 2$ and $y = 0$.



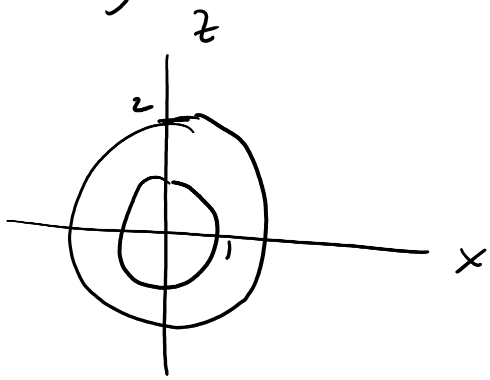
This is the plane $y = x+2$

project on the xz plane.

left: $y = 0$

Right: $y = x+2$

Region D



notice Region D is a polar region.

$1 \le r \le 2$

$0 \le \theta \le 2\pi$

$x = r\cos\theta \qquad z = r\sin\theta$

$x^2 + z^2 = r^2$

$$V = \iiint_E 1 \, dV = \iint_D \left[\int_{y=0}^{x+2} 1 \, dy \right] dA$$

$$= \iint_D y \Big|_0^{x+2} dA = \iint_D x+2 \, dA$$

now convert to polar.

$\int^{2\pi} \int^{2}$

$$\underbrace{\int_{\theta=0}^{2\pi} \int_{r=1}^{2} (r\cos\theta + 2)\, r \, dr \, d\theta}$$

Problem 6

Rewrite the integral $\int_0^1 \int_0^{2-2y} \int_0^{4-x^2} f(x,y,z)\, dz\,dx\,dy$ in the order of dy dx dz.

Top $z = 4 - x^2$

Bottom $z = 0$

Region D in the xy-plane

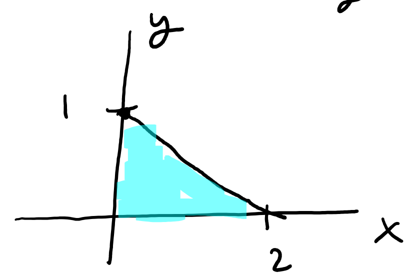
$0 \le x \le 2 - 2y$

$0 \le y \le 1$

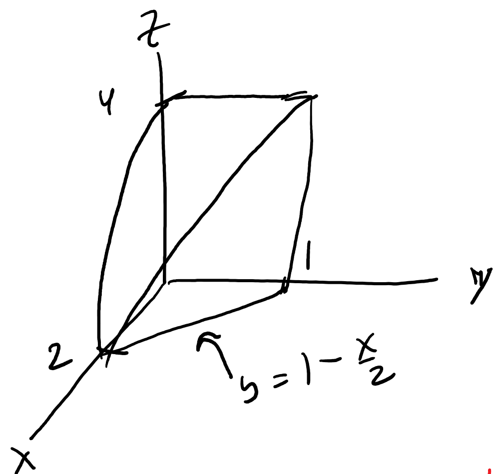
$x = 2 - 2y$

$2y = 2 - x$

$y = 1 - \frac{x}{2}$



our solid



Since we want

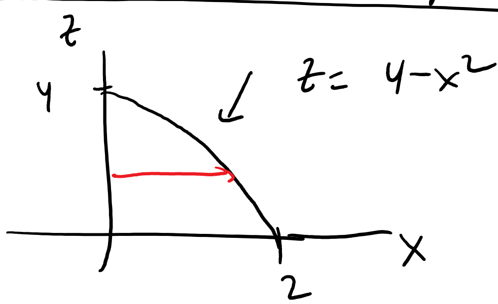
$dy\ dx\ dz$

Left: $y = 0$

Right: $y = 1 - \frac{x}{2}$

projection xz plane.

Region D (on xz plane)



$dx\ dz$

$0 \le z \le 4$

$0 \le x \le \sqrt{4-z}$

$x^2 = 4 - z$

$x = \pm\sqrt{4-z}$

Answer

$$\int_0^4 \int_0^{\sqrt{4-z}} \int_0^{1-\frac{x}{2}} f(x,y,z)\, dy\, dx\, dz$$

Answer

$$\boxed{\int_{z=0}^{4} \int_{x=0} \int_{y=0} f(x,y,z)\, dy\, dx\, dz}$$

Problem 7

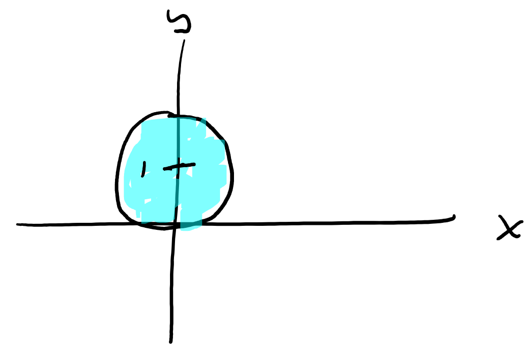
Set up the integral that will find the mass of a solid. The solid is inside the cylinder $x^2 + y^2 = 2y$, under the surface $z = 15 + 2x^2 + 2y^2$ and above the plane $z = 3y$. The density function of the solid is $\cancel{\rho(x,y) = y^2}$

$\rho(x,y,z) = y^2$

Top : $z = 15 + 2x^2 + 2y^2$

Bottom : $z = 3y$

projection xy plane



$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y = 0$$

$$x^2 + y^2 - 2y + (-1)^2 = (-1)^2$$

$$x^2 + (y-1)^2 = 1$$

$$x^2 = 1 - (y-1)^2$$

$$x = \pm\sqrt{1-(y-1)^2}$$

$$0 \le y \le 2$$

$$-\sqrt{1-(y-1)^2} \le x \le \sqrt{1-(y-1)^2}$$

OR

$$-\sqrt{2y-y^2} \le x \le \sqrt{2y-y^2}$$

$$\text{mass} = \iiint_E \rho(x,y,z)\, dV = \iiint_E y^2\, dV$$

$$= \int_{y=0}^{2} \int_{x=-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{z=3y}^{15+2y^2+2x^2} y^2 \, dz\, dx\, dy$$