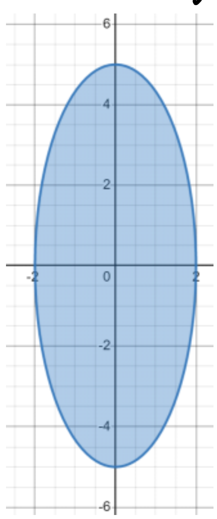
Problem 1

Use the given transformation to evaluate the integral.

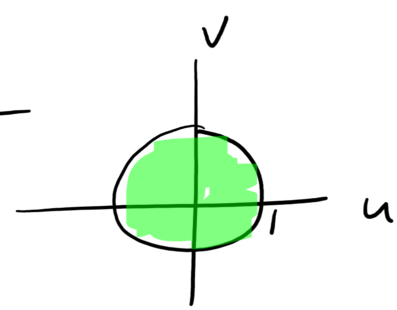
$\iint_R 3x^2 dA$, where R is the region bounded by the ellipse $25x^2 + 4y^2 \leq 100$; $x = 2u$, $y = 5v$.

Region R.



$$25x^2 + 4y^2 = 100$$
$$25(2u)^2 + 4(5v)^2 = 100$$
$$25 \cdot 4u^2 + 4 \cdot 25v^2 = 100$$
$$100u^2 + 100v^2 = 100$$
$$u^2 + v^2 = 1$$

Region S



$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} = 10$$

$$|J| = |10| = 10$$

we are looking to convert to polar.

$0 \leq r \leq 1 \quad 0 \leq \theta \leq 2\pi$

$$\iint_R 3x^2 \, dA = \iint_S 3(2u)^2 \cdot |J| \, dA = \iint_S 12u^2 \cdot 10 \, dA$$

$$= \iint_S 120u^2 \, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 120(r\cos\theta)^2 \cdot r \, dr \, d\theta$$

$S \qquad\qquad\qquad\qquad \theta=0 \quad r=0$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 120 r^3 \cos^2\theta \, dr \, d\theta$$

$$= 120 \int_{\theta=0}^{2\pi} \cos^2\theta \, d\theta \cdot \int_{r=0}^{1} r^3 \, dr$$ by Fubini

$$= 120 \int_{\theta=0}^{2\pi} \frac{1}{2}(1+\cos 2\theta) \, d\theta \cdot \left.\frac{r^4}{4}\right|_0^1$$

$$= 120 \cdot \frac{1}{2} \left[\theta + \frac{1}{2}\sin 2\theta\right]\Bigg|_{\theta=0}^{2\pi} \cdot \left(\frac{1^4}{4} - 0\right)$$

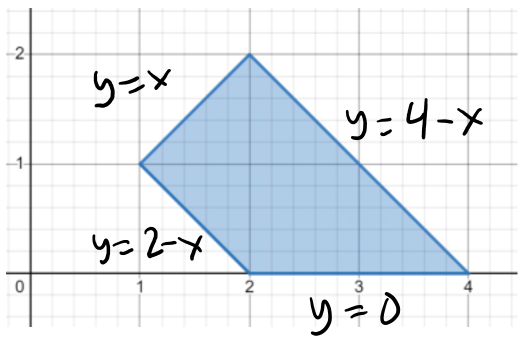
$$= 60 \cdot \left[2\pi + \frac{1}{2}\sin 4\pi - \left(0 + \frac{1}{2}\sin 0\right)\right] \cdot \frac{1}{4}$$

$$= 60 \left(2\pi + 0 - 0\right) \cdot \frac{1}{4} = 30\pi$$

Problem 2

Use the given transformation to evaluate the integral.

$\iint_R \sin\left(\frac{y-x}{y+x}\right) dA$, where R is the region bounded by the trapezoid with vertices $(1,1)$, $(2,2)$, $(4,0)$, $(2,0)$ and a change of variables: $u = y - x$, $v = y + x$



Lets solve for x+y.

$u = y - x$
$v = y + x$
$u + v = 2y$
$y = \frac{1}{2}(u+v)$
$y = \frac{1}{2}u + \frac{1}{2}v$

$u = y - x$
$-(v = y + x)$
$u - v = -2x$
$x = -\frac{1}{2}(u-v)$
$x = -\frac{1}{2}u + \frac{1}{2}v$

find Region S

$y = x$ | $y - x = 0$, $u = 0$

$y = 4 - x$ | $y + x = 4$, $v = 4$

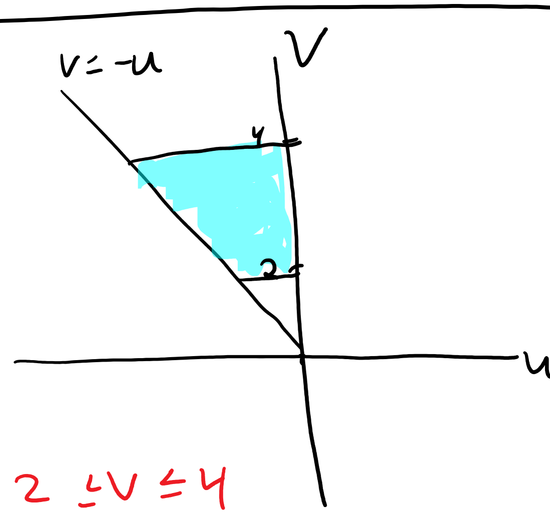
$y = 2 - x$ | $y + x = 2$, $v = 2$

$y = 0$ | $y = 0$
$\frac{1}{2}u + \frac{1}{2}v = 0$
$u + v = 0$
$u = -v$
or $v = -u$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$|J| = \left|-\frac{1}{2}\right| = \frac{1}{2}$$



$2 \le v \le 4$
$-v \le u \le 0$

$$\iint_R \sin\left(\frac{y-x}{y+x}\right) dA = \iint_S \sin\left(\frac{u}{v}\right) |J| \, dA$$

$$= \iint_S \sin\left(\frac{u}{v}\right) \cdot \frac{1}{2} \, dA = \int_{v=2}^{4} \int_{u=-v}^{0} \frac{1}{2} \sin\left(\frac{u}{v}\right) du\,dv$$

$$= \int_{v=2}^{4} -\frac{1}{2} v \cos\left(\frac{u}{v}\right) \Bigg|_{u=-v}^{0} dv$$

$$= \int_{v=2}^{4} -\frac{1}{2} v \cos(0) - -\frac{1}{2} v \cos(-1) \, dv$$

$$= \int_{v=2}^{4} -\frac{1}{2} v + \frac{1}{2} v \cos(-1) \, dv = -\frac{1}{4} v^2 + \frac{1}{4} v^2 \cos(-1) \Bigg|_{v=2}^{4}$$

$$= -\frac{1}{4}(16) + \frac{1}{4}(16)\cos(-1) - \left[-\frac{1}{4}(4) + \frac{1}{4}(4)\cos(-1)\right]$$

$$= -4 + 4\cos(-1) + 1 - \cos(-1)$$

$$= -3 + 3\cos(-1)$$