Problem 1

Evaluate $\int_C x^2yz\,ds$, where C is the line segment from $(1,0,2)$ to $(4,3,0)$.

$$r(t) = (1-t)\langle 1,0,2\rangle + t\langle 4,3,0\rangle$$

$$= \langle 1-t, 0, 2-2t\rangle + \langle 4t, 3t, 0\rangle$$

$$r(t) = \langle 1+3t, 3t, 2-2t\rangle$$

$$r' = \langle 3, 3, -2\rangle$$

$$|r'(t)| = \sqrt{9+9+4} = \sqrt{22}$$

$$\int_C x^2yz\,ds = \int_0^1 (1+3t)^2(3t)(2-2t)\sqrt{22}\,dt$$

$$= \sqrt{22}\int_0^1 (1+6t+9t^2)(6t-6t^2)\,dt$$

$$= \sqrt{22}\int_0^1 6t - 6t^2 + 36t^2 - 36t^3 + 54t^3 - 54t^4\,dt$$

$$= \sqrt{22}\int_0^1 6t + 30t^2 + 18t^3 - 54t^4\,dt$$

$$= \sqrt{22}\left[3t^2 + 10t^3 + \frac{9}{2}t^4 - \frac{54}{5}t^5\right]\Bigg|_0^1$$

$$= \sqrt{22}\left[3 + 10 + \frac{9}{2} - \frac{54}{5}\right] = 6.7\sqrt{22}$$

or $\frac{67}{10}\sqrt{22}$

Problem 2

Evaluate $\int_C y\,dx + (3x^2+y)\,dy$, where C is the curve consisting of the the arc of the curve $y = 9 + x^3$ from the point $(-1, 8)$ to $(2, 17)$ and then the line segment from the point $(2, 17)$ to the point $(4, 0)$

$C = C_1 + C_2$

C_1 $\quad x = t \quad y = 9 + t^3 \quad -1 \le t \le 2$

$$\int_{C_1} y\,dx + (3x^2+y)\,dy = \int_{-1}^{2} (9+t^3)\cdot 1 + (3t^2 + 9 + t^3)\cdot 3t^2\,dt$$

$$= \int_{-1}^{2} 9 + t^3 + 9t^4 + 27t^2 + 3t^5\,dt$$

$$= \left(9t + \frac{t^4}{4} + \frac{9t^5}{5} + 9t^3 + \frac{t^6}{2}\right)\Bigg|_{-1}^{2}$$

$$= 18 + 4 + \frac{9}{5}(32) + 9(8) + 32 - \left(-9 + \frac{1}{4} - \frac{9}{5} - 9 + \frac{1}{2}\right)$$

$$= 183.6 - \frac{-381}{20} = 202.65 = \frac{4053}{20}$$

C_2 $\quad$ line segment from $(2, 17)$ to $(4, 0)$

$$r(t) = (1-t)\langle 2, 17\rangle + t\langle 4, 0\rangle$$

$$= \langle 2-2t, 17-17t\rangle + \langle 4t, 0\rangle$$

$$= \langle 2+2t, 17-17t\rangle \qquad 0 \le t \le 1$$

$$\int_{C_2} y\,dx + (3x^2+y)\,dy = \int_0^1 (17-17t)\cdot 2 + \left[3(2+2t)^2 + 17-17t\right](-17)\,dt$$

$$= \int_0^1 17\cdot 2(1-t) + \left[3\left(4+8t+4t^2\right) + 17-17t\right](-17)\,dt$$

$$= 17\int_0^1 2-2t + \left(12+24t+12t^2+17-17t\right)(-1)\,dt$$

$$= 17\int_0^1 2-2t-12-24t-12t^2-17+17t\,dt$$

$$= 17\int_0^1 -27-9t-12t^2\,dt = 17\left[-27t - \frac{9t^2}{2} - 4t^3\right]\Bigg|_0^1$$

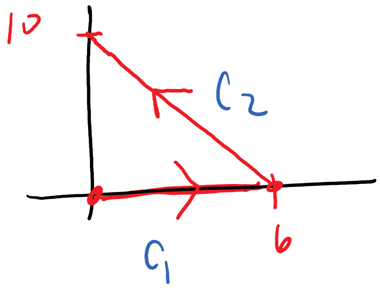
$$= 17\left(-27-\frac{9}{2}-4\right) - 0 = \frac{-1207}{2}$$

Answer

$$\frac{4053}{20} + \frac{-1207}{2} = -400.85 = \frac{-8017}{20}$$

Problem 3

Evaluate $\oint_C (x+y)dx + (2x+y)dy$, where C is the path from the point $(0,0)$ to $(6,0)$ to $(0,10)$



C_1 | $r(t) = (1-t)\langle 0,0\rangle + t\langle 6,0\rangle$

$r(t) = \langle 6t, 0\rangle$

C_2 | $r(t) = (1-t)\langle 6,0\rangle + t\langle 0,10\rangle$

$r(t) = \langle 6-6t, 10t\rangle$

$$\int_{C_1} (x+y)dx + (2x+y)dy = \int_0^1 6t \cdot 6 + 0 \, dt$$

$$= \int_0^1 36t \, dt = 18t^2 \Big|_0^1 = 18$$

$$\int_{C_2} (x+y)dx + (2x+y)dy = \int_0^1 (6-6t+10t)\cdot(-6) + (12-12t+10t)10 \, dt$$

$$= \int_0^1 (6+4t)(-6) + (12-2t)10 \, dt$$

$$= \int_0^1 -36-24t+120-20t \, dt$$

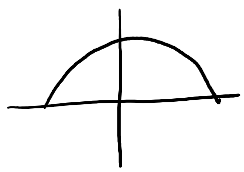
$$= \int_0^1 84-44t \, dt = (84t-22t^2)\Big|_0^1$$

$$= 84 - 22 = 62$$

answer $18 + 62 = 80$

Problem 4

A thin wire with linear density $\rho(x, y) = 2 + x^2 y$ takes the shape of the semicircle $x^2 + y^2 = 4$, $y \geq 0$. Find the center of mass for this wire.



$C: \quad x = 2\cos\theta \qquad 0 \leq \theta \leq \pi$

$\quad\;\; y = 2\sin\theta \qquad ds = \sqrt{(-2\sin\theta)^2 + (2\cos\theta)^2} = \sqrt{4} = 2$

mass

$$m = \int_C (2 + x^2 y)\, ds = \int_0^\pi \left(2 + 4\cos^2\theta \cdot 2\sin\theta\right) 2\, d\theta$$

$$= \int_0^\pi 4 + 16\cos^2\theta \sin\theta \, d\theta$$

$$= \left(4\theta - \frac{16}{3}\cos^3\theta\right)\Big|_0^\pi$$

$$= 4\pi - \frac{16}{3}(\cos\pi)^3 - \left(0 - \frac{16}{3}\right)$$

$$= 4\pi + \frac{16}{3} + \frac{16}{3} = \underline{4\pi + \frac{32}{3}} = m \text{ (mass}$$

$$\bar{x} = \frac{1}{m}\int_C x\,\sigma(x, y)\, ds = \frac{1}{m}\int_C x(2 + x^2 y)\, ds$$

$$= \frac{1}{m}\int_0^\pi \left(2 \cdot 2\cos\theta + 2^3\cos^3\theta \; 2\sin\theta\right) \cdot 2\, d\theta$$

$$= \frac{1}{m}\int_0^{\pi} 8\cos\theta + 32\cos^3\theta\sin\theta \, d\theta$$

$$= \frac{1}{m}\left[8\sin\theta - \frac{32}{4}\cos^4(\theta) \right]_0^{\pi}$$

$$= \frac{1}{m}\left[\left(0 - \frac{32}{4}(-1)^4\right) - \left(0 - \frac{32}{4}(1)^4\right) \right]$$

$$= \frac{1}{m}\left[-\frac{32}{4} + \frac{32}{4} \right] = 0 = \bar{x}$$

$$\bar{y} = \frac{1}{m}\int_C y\,\sigma(x,y)\,ds = \frac{1}{m}\int_C y\,(2 + x^2 y)\,ds$$

$$= \frac{1}{m}\int_0^{\pi} 2\sin\theta\left(2 + 4\cos^2\theta\, 2\sin\theta\right)\cdot 2\, d\theta$$

$$= \frac{1}{m}\int_0^{\pi} 8\sin\theta + 32\cos^2\theta\sin^2\theta \, d\theta$$

$$= \frac{1}{m}\int_0^{\pi} 8\sin\theta + 32\cdot\frac{1}{2}(1+\cos 2\theta)\cdot\frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{m} \int_0^{\pi} 8\sin\theta + 8\left(1 - \cos^2 2\theta\right)\, d\theta$$

$$= \frac{1}{m} \int_0^{\pi} 8\sin\theta + 8 - 8 \cdot \frac{1}{2}\left(1 + \cos 4\theta\right)\, d\theta$$

$$= \frac{1}{m} \int_0^{\pi} 8\sin\theta + 8 - 4 - 4\cos 4\theta\, d\theta$$

$$= \frac{1}{m} \int_0^{\pi} 8\sin\theta + 4 - 4\cos 4\theta\, d\theta$$

$$= \frac{1}{m} \left[-8\cos\theta + 4\theta - 4 \cdot \frac{1}{4}\sin 4\theta \right]\Big|_0^{\pi}$$

$$= \frac{1}{m} \left[-8(-1) + 4\pi - \sin(4\pi) - \left(-8 + 0 - \sin(0)\right)\right]$$

$$= \frac{1}{m} \left[8 + 4\pi + 8 \right] = \frac{1}{m} \left[16 + 4\pi \right]$$

$$= \frac{1}{4\pi + \frac{32}{3}} \cdot \left(16 + 4\pi\right)$$

Center of mass $\left(0, \dfrac{16+4\pi}{4\pi + \frac{32}{3}} \right)$