Problem 1

Example: Let S be the sphere $x^2 + y^2 + z^2 = 16$ with a positive orientation and $\mathbf{F} = \langle 0, 0, z \rangle$. Use the Divergence Theorem to evaluate $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$

$$\operatorname{div} F = 0 + 0 + 1 = 1$$

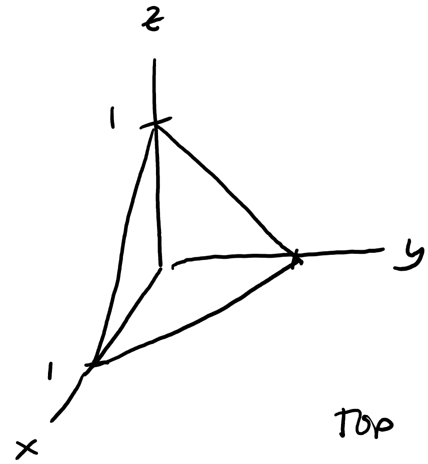
$$\iint\limits_S F \cdot dS = \iiint\limits_E \operatorname{div} F \, dV = \iiint\limits_E 1 \, dV = \text{volume of solid } E$$

$$= \frac{4}{3}\pi (4)^3 = \frac{256\pi}{3}$$

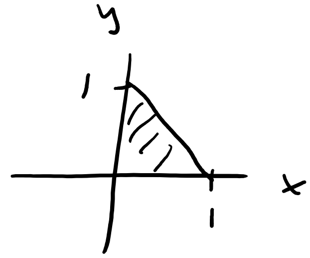
Problem 2

Let S be the closed surface of a tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e .the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F} = \langle y, z-y, x\rangle$ and use positive orientation.

Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$



$$= \iiint_E 0 + (-1) + 0 \, dV$$



Top $z = 1-x-y$

Bottom $z=0$

Region D

$0 \le x \le 1$

$0 \le y < 1-x$

$$= \iiint_E -1 \, dV =$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} -1 \, dz\, dy\, dx = \int_{x=0}^{1}\int_{y=0}^{1-x} -z \Big|_0^{1-x-y} dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} -1 + x + y \, dy\, dx = \int_{x=0}^{1} -y + xy + \frac{y^2}{2} \Big|_0^{1-x} dx$$

$$= \int_{x=0}^{1} -(1-x) + x(1-x) + \frac{1}{2}(1-x)^2 \, dx$$

$$= \int_{x=0}^{1} -1 + x + x - x^2 + \frac{1}{2}(1-x)^2 \, dx = \int_{x=0}^{1} -1 + 2x - x^2 + \frac{1}{2}(1-x)^2 \, dx$$

$x=0$ $x=0$

$$= -x + x^2 - \frac{x^3}{3} + \frac{1}{2}\left(\frac{1}{3}\right)(-1)(1-x)^3 \Big|_0^1$$

$$= -1 + 1 - \frac{1}{3} - \frac{1}{6}(0)^3 - \left(-0 + 0 - 0 - \frac{1}{6}(1)^3\right)$$

$$= -\frac{1}{3} + \frac{1}{6} = -\frac{1}{6}$$

Problem 3

Calculate the flux of $\mathbf{F}$ across the surface S with the equation $x^4 + y^4 + z^4 = 1$. Assume positive orientation for the surface.

$\mathbf{F} = \langle x + e^{y\tan(z)},\ 3xe^{xz},\ \cos(y) - z \rangle$

$$\text{div } F = 1 + 0 + (-1) = 0$$

$$\iint_S F \cdot dS = \iiint_E \text{div } F \cdot dV = \iiint_E 0 \, dV = 0$$

Fun Question

Fun Question: This question is for those of you who find this interesting. This is not an exam question.

Compute $\iint_S (3x + 8y + z^2)dS$ Where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$

lets look at $\iint_S F \cdot dS = \iint_S F \cdot n \, ds = \iint_D F \cdot (r_u \times r_v) \, dA$

now if $\iint_S F \cdot n \, dS = \iint_S 3x + 8y + z^2 \, ds$

Then $F \cdot n = 3x + 8y + z^2$ where n is a unit normal vector of the surface S (positive orientation)

for the sphere $\underbrace{x^2 + y^2 + z^2}_{G(x,y,z)} = 1$ we find a normal vector for any tangent plane is $\nabla G = \langle G_x, G_y, G_z \rangle$

$= \langle 2x, 2y, 2z \rangle$

we can use any normal vector that points out from the sphere so lets use $\langle x, y, z \rangle$

unit normal $n = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} = \langle x, y, z \rangle$ since $x^2 + y^2 + z^2 = 1$ (sphere formula)

Since $F \cdot n = 3x + 8y + z^2$

This means $\mathbf{F} = \langle 3, 8, z \rangle$

---

now $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 1 \, dV$

$$= \frac{4}{3}\pi (1)^3 = \frac{4\pi}{3}$$