4 Second-order linear equations with constant coefficients

4.1 A homogeneous, second-order, linear equation with constant coefficients has the (normal) form

$$y'' + py' + qy = 0 \tag{4.1}$$

If there is a constant factor in front of the term y'', divide by this factor to bring the equation into normal form.

Solution of homogeneous, 2nd-order, linear ODE with constant coefficients

In order to solve equations of type (4.1), first solve the auxiliary equation

$$r^2 + pr + q = 0.$$

Depending on the discriminant $D = p^2 - 4q$, there are three possibilities:

1. $D > 0$: The auxiliary equation has two real solutions r_1 and r_2. From these we obtain the two linearly independent solutions

$$y_1 = e^{r_1 x} \qquad\qquad y_2 = e^{r_2 x}.$$

2. $D < 0$: The auxiliary equation has two conjugate complex solutions $r \pm i\alpha$. From these we obtain the two linearly independent solutions

$$y_1 = e^{rx} \sin \alpha x \qquad\qquad y_2 = e^{rx} \cos \alpha x.$$

3. $D = 0$: The auxiliary equation only has a single real solution r. From this we obtain the two linearly independent solutions

$$y_1 = e^{rx} \qquad\qquad y_2 = xe^{rx}.$$

The fundamental solution of (4.1) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

4.2 An inhomogeneous, second-order, linear equation with constant coefficients has the form

$$y'' + py' + qy = f \tag{4.2}$$

Solution of inhomogeneous, 2nd order, linear ODE with constant coefficients

1. Compute the fundamental solution of the homogeneous equation (4.1).
2. Compute one *particular* solution y_p of the inhomogeneous equation (4.2).
3. The fundamental solution of the inhomogeneous equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

4.3 If the right hand sided of the equation is simple, a particular solution to an inhomogeneous, 2nd-order, linear equation can be guessed using the following method:

Undetermined coefficient

1. Compute the linearly independent solutions y_1 and y_2 according to 4.1.
2. Depending on the right hand side of the ODE, choose a guess from the following table:

right hand side	**guess**
$e^{\alpha t}$	$a e^{\alpha t}$
$\alpha \sin \nu t + \beta \cos \nu t$	$a \sin \nu t + b \cos \nu t$
$\alpha_n t^n + \alpha_{n-1} t^{n-1} + \cdots + \alpha_1 t + \alpha_0$	$a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$

   Make sure that you use the full trigonometric or polynomial expansion, even if you only have a single sine, cosine or monomial.

3. Check, if your guess is a solution to the homogeneous equation. If so, multiply with t. Repeat, until your guess is not a solution to the homogeneous equation.
4. Insert your guess into the inhomogeneous equation and solve for the coefficients.

4.4 A particular solution to an inhomogeneous, 2nd-order, linear equation can be computed using the following method:

Variation of parameters

1. Compute the linearly independent solutions y_1 and y_2 according to 4.1.
2. Obtain functions u_1 and u_2 by solving the system

$$\begin{array}{ccccc} u_1' y_1 & + & u_2' y_2 & = & 0 \\ u_1' y_1' & + & u_2' y_2' & = & f \end{array} \tag{4.3}$$

   and integrating u_1' and u_2' (intgration constants are of no importance).

3. A particular solution is now obtained as

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x).$$

4.5 Note that the first equation in (4.3) has no dependence on the right hand side. Even more, in the case of complex solutions to the characteristic equations, it always gives $u_1' = -\tan(\beta x)\, u_2'$.